around 1955?

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(1) $\omega(t) = \omega(|t|)$

(2) $\omega(t)$ is monotone decreasing for $t > 0$

(3) $N_1(\omega) = \omega(0) + \int_{-\infty}^{\infty} \omega(t)dt < \infty$.

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(1) $$\inf_{\omega \in \Omega_1} N_1(\omega) \int_{-\infty}^{\infty} \frac{|\varphi(t)|^2 dt}{\omega} < \infty .$$

The quantity on the left of equation (1) is defined as the square norm on A_1^2, $||\varphi||^2_{A_1^2}$. It is proved in [2] that $||\varphi||_{A_1^2}$ is indeed a norm.

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The Integral Equation $\int_0^\infty \rho(t-s)f(s)ds = \lambda f(t), \quad t > 0.$

The Spaces A_1^2 and B_1^2

We begin the study of this homogeneous integral equation with a summary of some properties of the spaces A_1^2 and B_1^2 in which the kernel and the solution will respectively be assumed to lie. A general theory of these spaces in the framework of convolution algebras may be found in [2]. All these function spaces are in our case over the doubly-infinite real line $(-\infty, \infty)$.

Definitions: A function $\omega(t)$ is a member of Ω_1 if

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A function $\varphi(t)$ is a member of A_1^2 if

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A function $f(t)$ is a member of B_1^2 if

$$(2) \qquad \sup_{a>0} \frac{1}{1+2a} \int_{-a}^{a} |f(t)|^2 dt < \infty .$$

The quantity on the left of (2) is defined as the square norm on B_1^2, $||f||^2_{B_1^2}$. It is easily seen that $||f||_{B_1^2}$ is a norm.

We shall need a few elementary facts regarding the spaces A_1^2 and B_1^2.

a) If $\rho(t) \in A_1^2$, then $\rho(t) \in L^1$.

If $f(t) \in L^\infty$, then $f(t) \in B_1^2$.

The second of these is obvious. To prove the first, pick an $\omega \in \Omega_1$ such that (1) is finite. Then

$$\left[\int_{-\infty}^{\infty} |\rho(t)| dt \right]^2 \leq \int_{-\infty}^{\infty} \frac{|\rho(t)|^2 dt}{\omega(t)} \int_{-\infty}^{\infty} \omega(t) dt$$

$$< ||\rho||^2_{A_1^2} < \infty .$$

b) If $f(t) \in B_1^2$, then $f(t+\alpha) \in B_1^2$, and

$$||f(t+\alpha)||_{B_1^2} \leq (1+|\alpha|)^{\frac{1}{2}} ||f(t)||_{B_1^2} .$$

For, taking first $\alpha > 0$,

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$$\frac{1}{1+2a}\int_{-a}^{a}|f(t+\alpha)|^2dt \le \frac{1}{1+2a}\int_{-(a+\alpha)}^{a+\alpha}|f(t)|^2dt$$

$$= \frac{1+2(a+\alpha)}{1+2a}\ \frac{1}{1+2(a+\alpha)}\int_{-(a+\alpha)}^{a+\alpha}|f(t)|^2dt$$

$$\le \sup_a \frac{1+2(a+\alpha)}{1+2a}\cdot ||f||^2_{B_1^2}$$

$$= (1+\alpha)||f||^2_{B_1^2}.$$

Hence $||f(t+\alpha)||_{B_1^2} \le (1+\alpha)^{\frac{1}{2}}\ ||f||_{B_1^2}$, and a similar argument for $\alpha < 0$ gives our result.

c) An equivalent definition of B_1^2 and its norm is given as follows:

$$(3)\quad \sup_{a>0}\frac{1}{1+2a}\int_{-a}^{a}|f(t)|^2dt = \sup_{\substack{\omega\ \varepsilon\ \Omega_1 \\ N_1(\omega)=1}}\int_{-\infty}^{\infty}|f(t)|^2\omega(t)dt\ .$$

Proof: Pick $\omega(t) = \begin{cases} \frac{1}{1+2a} & |t| < a \\ 0 & |t| > a\ . \end{cases}$

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Then $N_1(\omega) = 1$, and we obtain the left-hand side of (3). This proves

$$\sup_{a>0} \frac{1}{1+2a} \int_{-a}^{a} |f(t)|^2 dt \leq \sup_{\substack{\omega \in \Omega_1 \\ N_1(\omega)=1}} \int_{-\infty}^{\infty} |f(t)|^2 \omega(t) dt .$$

On the other hand, for $\omega \in \Omega_1$,

$$\int_{-A}^{A} |f(t)|^2 \omega(t) dt = \int_{0}^{A} [|f(t)|^2 + |f(-t)|^2] \omega(t) dt$$

$$= \omega(A) \int_{-A}^{A} |f(s)|^2 ds + \int_{0}^{A} \left[\int_{-t}^{t} |f(s)|^2 ds \right] d(-\omega(t))$$

$$\leq ||f||^2_{B_1^2} \left[(1+2A)\omega(A) + \int_{0}^{A} (1+2t) d(-\omega(t)) \right]$$

$$= ||f||^2_{B_1^2} \left[\omega(0) + \int_{-A}^{A} \omega(t) dt \right] \leq ||f||^2_{B_1^2} .$$

Hence, for any $\omega \in \Omega_1$,

$$\int_{-\infty}^{\infty} |f(t)|^2 \omega(t) dt \leq \sup \frac{1}{1+2a} \int_{-a}^{a} |f(t)|^2 dt .$$

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This proves the equality in (3).

d) If $\rho \in A_1^2$ and $f \in B_1^2$, then

$$m(t) = \int_{-\infty}^{\infty} \rho(t-s)f(s)ds$$

is in B_1^2 and

$$||m||_{B_1^2} \leq ||\rho||_{A_1^2} ||f||_{B_1^2} .$$

The full proof of this theorem requires a proof that Ω_1 is a convolution ring, and may be found in [2]. We may, however, give a simple proof of the first part in the special case that also $\rho(t)[1+|t|]^{\frac{1}{2}} \in A_1^2$. In this case,

$$|m(t)|^2 \leq \int_{-\infty}^{\infty} \frac{|\rho(s)|^2(1+|s|)}{\omega(s)} ds \int_{-\infty}^{\infty} \frac{|f(t-s)|^2\omega(s)}{1+|s|} ds,$$

where $\omega \in \Omega_1$ is chosen so that the first integral is finite. Then pick $\omega_1 \in \Omega_1$.

$$\int_{-\infty}^{\infty} \omega_1(t)|m(t)|^2dt \leq C \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|f(t-s)|^2\omega(s)\omega_1(t)}{1+|s|} ds\,dt .$$

But by c) and then b) above,

$$\int_{-\infty}^{\infty} |f(t-s)|^2 \omega_1(t)dt \leq c_1(1+|s|) ,$$

so that

$$\int_{-\infty}^{\infty} \omega_1(t)|m(t)|^2 dt \leq c_2 \int_{-\infty}^{\infty} \omega(s)ds < c_2 .$$

This proves that

$$\sup_{\substack{\omega_1 \in \Omega_1 \\ N_1(\omega_1) = 1}} \int_{-\infty}^{\infty} \omega_1(t)|m(t)|^2 dt < \infty,$$

and hence $m(t) \in B_1^2$.

The Convolution Lemma

If $k(y) \in A_1^2$, then, as was shown in a), $k(y) \in L^1$, and hence $k(y)$ has a Fourier transform $K(t)$.

If $\varphi(y) \in B_1^2$, then $\varphi(y)e^{-\sigma|y|} \in L^1$ for any $\sigma > 0$ since $e^{-\sigma|y|} \in \Omega_1$. Hence $\varphi(y)e^{-\sigma|y|}$ has a Fourier transform, which will be denoted by $\Phi(\sigma,t)$. $\Phi(\sigma,t)$ is obviously harmonic in the right-half (σ,t)-plane, and has also been called the harmonic transform of $\varphi(y)$.

Lemma: Let

$$(4) \qquad \psi(x) = \int_{-\infty}^{\infty} \varphi(y)k(x-y)dy \ ,$$

where φ and ψ are in B_1^2, and $(1+|y|)^{\frac{1}{2}}k(y) \in A_1^2$. If (4) is satisfied, then

$$(5) \qquad \lim_{\sigma \to 0^+} \frac{1}{2\pi} \int_{-\infty}^{\infty} |\Phi(\sigma,t)K(t)-\Psi(\sigma,t)|^2 dt = 0,$$

where Φ, Ψ, and K are respectively the Fourier transforms of $\varphi(y)e^{-\sigma|y|}$, $\psi(y)e^{-\sigma|y|}$, and $k(y)$. Conversely, if for any finite a and b,

$$(6) \qquad \lim_{\sigma \to 0^+} \frac{1}{2\pi} \int_{a}^{b} |\Phi(\sigma,t)K(t)-\Psi(\sigma,t)|^2 dt = 0,$$

then (4) is satisfied.

An analogous lemma for L^1 and L^∞ may be found in [1].

Proof: Let

$$\psi_\sigma(x) = \int_{-\infty}^{\infty} e^{-\sigma|x-y|} \varphi(x-y)k(y)dy \ .$$

Then the Fourier Transform of $\psi_\sigma(x)$ is $\Phi(\sigma,t)K(t)$. If $\delta_\sigma(x) = \psi_\sigma(x) - e^{-\sigma|x|}\psi(x)$, then

$$\delta_\sigma(x) = \int_{-\infty}^{\infty} \left\{ e^{-\sigma|x-y|} - e^{-\sigma|x|} \right\} \varphi(x-y)k(y)dy \ .$$

Thus

$$|\delta_\sigma(x)|^2 \leq \int_{-\infty}^{\infty} \frac{|k(y)|^2(1+|y|)}{\omega(y)} dy \int_{-\infty}^{\infty} \left\{ e^{-\sigma|x-y|} - e^{-\sigma|x|} \right\}^2 \frac{|\varphi(x-y)|^2}{1+|y|} \omega(y)dy,$$

where $\omega \in \Omega_1$ is chosen so that the first integral is finite.

Therefore

$$\int_{-\infty}^{\infty} |\delta_\sigma(x)|^2 dx \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ e^{-\sigma|x-y|} - e^{-\sigma|x|} \right\}^2 \frac{|\varphi(x-y)|^2}{1+|y|} \omega(y)dy\,dx$$

$$= \int_{-\infty}^{\infty} \frac{\omega(y)dy}{1+|y|} \int_{-\infty}^{\infty} \left\{ e^{-\sigma|\xi|} - e^{-\sigma|\xi+y|} \right\}^2 |\varphi(\xi)|^2 d\xi$$

$$\leq \int_{-\infty}^{\infty} \frac{\omega(y)dy}{1+|y|} \inf_{\substack{\omega^* \in \Omega_1 \\ \omega^* \geq \left\{ e^{-\sigma|\xi|} - e^{-\sigma|\xi+y|} \right\}^2}} N_1(\omega^*) \, ||\varphi||^2_{B_1^2} \ .$$

We may pick as an ω^* the function

$$\omega_o^* = \begin{cases} [1-e^{-\sigma|y|}]^2 & |\xi| \le |y| \\ [e^{-\sigma|\xi|}\ e^{-\sigma(|\xi|-|y|)}]^2 & |\xi| > |y| \end{cases}$$

Then

$$N_1(\omega_o^*) = (1+2|y|)(1-e^{-\sigma|y|})^2 + 2\int_{|y|}^{\infty}\left[e^{\sigma|y|}e^{-\sigma|\xi|} - e^{-\sigma|\xi|}\right]^2 d\xi$$

$$= (1+2|y|)(1-e^{-\sigma|y|})^2 + \frac{2}{\sigma}\left[\frac{1}{2} - e^{-\sigma|y|} + \frac{e^{-2\sigma|y|}}{2}\right]$$

$$= (1+2|y| + \tfrac{1}{\sigma})(1-e^{-\sigma|y|})^2 .$$

Thus

$$(7)\qquad \int_{-\infty}^{\infty}|\delta_\sigma(x)|^2dx \le c_1\int_{-\infty}^{\infty}\frac{\omega(y)dy}{1+|y|}(1+2|y| + \tfrac{1}{\sigma})(1-e^{-\sigma|y|})^2 .$$

We have

$$\frac{1+2|y|}{1+|y|}(1-e^{-\sigma|y|})^2 \le 2, \quad \text{and}$$

$$\frac{1-e^{-\sigma|y|}}{\sigma}\ \frac{1-e^{-\sigma|y|}}{1+|y|} \le |y|\ \ \frac{1-e^{-\sigma|y|}}{1+|y|} \le 1.$$

We see that both left-hand sides approach zero <u>boundedly</u> with σ for fixed y, and so does their sum, which is the coefficient of $\omega(y)$ in (7). Hence, by dominated convergence,

$$\lim_{\sigma \to 0^+} \int_{-\infty}^{\infty} |\delta_\sigma(x)|^2 dx = 0,$$

and by Parseval's and Plancherel's theorems, (5) now follows.

Now suppose (6) is satisfied for any finite a and b. Let

$$(8) \qquad \psi_1(x) = \int_{-\infty}^{\infty} \varphi(y) k(x-y) dy \ .$$

Since $\varphi \in B_1^2$, $k \in A_1^2$, it follows by d) that $\psi_1 \in B_1^2$. Hence the first part of the theorem applies, and

$$\lim_{\sigma \to 0^+} \frac{1}{2\pi} \int_a^b |\Phi(\sigma,t)K(t) - \Psi_1(\sigma,t)|^2 dt = 0$$

for any finite a and b. Hence, by the triangle inequality,

$$\lim_{\sigma \to 0^+} \int_a^b |\Psi(\sigma,t) - \Psi_1(\sigma,t)|^2 dt = 0 \ ,$$

and, by the Schwarz Inequality,

$$\lim_{\sigma \to 0^+} \int_a^b |\Psi(\sigma,t)-\Psi_1(\sigma,t)|\,dt = 0$$

as well.

Now let $\psi(x)-\psi_1(x) = h(x) \in B_1^2$, and let $H(\sigma,t) = \Psi(\sigma,t)-\Psi_1(\sigma,t)$ be the Fourier transform of $h(x)e^{-\sigma|x|}$. Then, for $\sigma > 0$,

$$|H(\sigma,t)| \leq \int_{-\infty}^{\infty} e^{-\sigma|y|}\,|h(y)|\,dy$$

$$\leq \left(\int_{-\infty}^{\infty} e^{-\sigma|y|}\,dy \int_{-\infty}^{\infty} |h(y)|^2 e^{-\sigma|y|}\,dy \right)^{\frac{1}{2}} \tag{9}$$

$$\leq \left(\frac{2}{|\sigma|}\,||h||^2_{B_1^2}\, N_1(e^{-\sigma|y|}) \right)^{\frac{1}{2}} \leq ||h||_{B_1^2} \left(1 + \frac{2}{|\sigma|}\right),$$

and

$$\lim_{\sigma \to 0^+} \int_a^b |H(\sigma,t)|\,dt = 0 \tag{10}$$

for fixed a and b. If $H(\sigma,t)$ is defined as $-H(-\sigma,t)$ for negative σ, we see that $H(\sigma,t)$ is a harmonic function in the whole plane, and hence the imaginary part of an entire function. Thus

$$H(\sigma,t) = \sum_{n=0}^{\infty} a_n r^n \sin n\theta,$$

and

$$a_n r^n = \frac{1}{\pi} \int_{-\pi}^{\pi} H(\sigma,t) \sin n\theta \, d\theta ,$$

where $\sigma + it = re^{i\theta}$. Then, by (9)

$$|a_n| \leq \frac{c}{\pi r^n} \int_{-\pi}^{\pi} |\sin n\theta| \left(1 + \frac{2}{r|\sin \theta|}\right) d\theta ,$$

and by taking r arbitrarily large, it follows that $a_n = 0$ if $n > 0$. Hence $H(\sigma,t)$ is a constant, and by (10), $H(\sigma,t) \equiv 0$. Thus $h(t) \equiv 0$, and $\psi_1(t) = \psi(t)$.

The Integral Equation

We proceed now to the solution of the integral equation

$$\int_0^{\infty} \rho(t-s) f(s) ds = \lambda f(t), \qquad t > 0, \quad \lambda \neq 0. \tag{11}$$

We make the following four assumptions:

(i). $(1 + |t|)^{\frac{1}{2}} \rho(t) \in A_1^2$.

By a) above, this implies $\rho(t) \in L^1$. Hence its Fourier Transform

$$r(x) = \int_{-\infty}^{\infty} \rho(t) e^{-itx}\, dt \tag{12}$$

is continuous and tends to 0 as $x \to \pm\infty$.

(ii). $\rho(-t) = \overline{\rho(t)}$. This means that $r(x)$ is real.

(iii). $r(x)-\lambda$ is to vanish at a finite number of points, and for any zero a of $r(x)-\lambda$, we assume that $r-\lambda$ changes signs and

$$\left| \frac{r(x)-\lambda}{x-a} \right| > \delta > 0 \quad \text{if} \quad 0 < |x-a| < \varepsilon. \tag{13}$$

Such a zero
~~A zero satisfying (13)~~ shall be called simple.

It follows from the remarks after (i) that $r(x)$ is bounded, and min $r(x)$ and max $r(x)$ shall be denoted by α and β respectively. The number of zeros of $r(x)-\lambda$ is even, and they shall be denoted by $\{a_\nu\}_{\nu=1}^{2n}$, $a_1 < a_2 < \cdots < a_{2n}$.

(iv). We shall seek solutions $f(t) \in B_1^2$.

Theorem. Under assumptions (i)-(iv), the number of linearly independent solutions of (11) is n, and the Fourier Transform, for Im $z < 0$, of each $f(t)$ is given by

$$\int_0^{\infty} f(t) e^{-itz}\, dt = \frac{\sum_{\nu=0}^{n-1} c_\nu z^\nu}{Q_2(z)}, \tag{14}$$

where $Q_2(z)$ will be defined below.

Proof: We define $f(t) = 0$ for $t < 0$, and we define

$$g(t) = \int_0^\infty \rho(t-s)f(s)ds \qquad t < 0$$

$$= 0 \qquad t > 0 . \tag{15}$$

We proceed to establish the properties of the Fourier Transforms of $f(t)$ and $g(t)$ for complex argument. Since $f(t) = 0$ for $t < 0$, we consider

$$F(x-iy) = \int_0^\infty f(t)e^{-it(x-iy)}dt \tag{16}$$

for $y > 0$. Then

$$|F(x-iy)| \le \int_0^\infty |f(t)|e^{-ty}dt$$

$$\le \left[\int_0^\infty |f(t)|^2 e^{-ty}dt \int_0^\infty e^{-ty}dt\right]^{\frac{1}{2}}$$

$$< \left(\frac{1}{y} N_1(e^{-|t|y}) \int_0^\infty |f(t)|^2 \frac{e^{-ty}}{N_1(e^{-ty})} dt\right)^{\frac{1}{2}}$$

$$\le \left[\frac{c}{y}\left(1 + \frac{1}{y}\right)\right]^{\frac{1}{2}}$$

Thus

$$|F(x-iy)| \leq \begin{cases} \frac{c}{|y|} & \text{as } y \to 0 \\ \frac{c}{|y|^{\frac{1}{2}}} & \text{as } y \to \infty . \end{cases} \tag{17}$$

For the function $g(t)$ we have

$$G(x+iy) = \int_{-\infty}^{0} g(t)e^{-it(x+iy)}dt \tag{18}$$

if $y > 0$. Now

$$g(t) = \int_{0}^{\infty} \rho(t-s)f(s)ds, \qquad t < 0 ,$$

and we note that $g(t) \in B_1^2$. We have

$$(1 + |t|)^{\frac{1}{2}}g(t) = \int_{0}^{\infty} \rho(t-s)(1+|t|)^{\frac{1}{2}}f(s)ds ,$$

and

$$(1 + |t|)^{\frac{1}{2}}g(t) \leq \int_{0}^{\infty} \rho(t-s)(1+|t-s|)^{\frac{1}{2}}f(s)ds ,$$

since $t < 0$ and $s > 0$.

Thus

$$(1 + |t|)^{\frac{1}{2}} g(t) \leq ||\rho(1+|t|)^{\frac{1}{2}}||_{A_1^2} \, ||f||_{B_1^2} = c$$

by d) above, and

$$|g(t)| \leq \frac{\varepsilon}{(1 + |t|)^{\frac{1}{2}}} .$$

Hence

$$|G(x+iy)| < c \int_{-\infty}^{0} \frac{e^{-|t|y} \, dt}{(1+|t|)^{\frac{1}{2}}} = c \, \frac{e^{y}}{\sqrt{y}} \int_{y}^{\infty} e^{-z} \, \frac{dz}{\sqrt{z}} .$$

We see that

$$(19) \qquad |G(x+iy)| \leq \begin{cases} \frac{c_1}{\sqrt{y}} & \text{as } y \to 0 \\ \frac{c_2}{y} & \text{as } y \to \infty . \end{cases}$$

The convolution lemma proved above, when applied to the combination of (11) and (15), yields that

$$(20) \qquad \lim_{y \to 0^+} \int_{-\infty}^{\infty} |[r(x)-\lambda]F(x-iy)-G(x+iy)|^2 dx = 0 .$$

The growth and regularity of F and G have just been discussed, and we proceed to the study of $r(x) - \lambda$.

Let $u(z)$ be the harmonic function in the lower half plane with boundary values $\log|r(x)-\lambda|$, i.e. the Poisson Integral

$$u(z) = -\frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\log|r(\xi)-\lambda|\,d\xi}{(\xi-x)^2+y^2} \,. \tag{21}$$

Let $v(z)$ be the harmonic conjugate of $u(z)$, and let

$$K_2(z) = e^{\frac{1}{2}(u+iv)} \,. \tag{22}$$

In the upper half plane, we define

$$K_1(z) = \frac{1}{\overline{K_2(\bar{z})}} = e^{-\frac{1}{2}u(x_1-y)+\frac{1}{2}iv(x_1-y)} \,. \tag{23}$$

Then

$$\lim_{y\to 0^+} \frac{K_2(x+iy)}{K_1(x+iy)} = |r(x)-\lambda)| \,, \tag{24}$$

and it remains to account for $\operatorname{sgn}(r(x)-\lambda)$. This can be done as follows: Let the complex plane be cut by n line segments from $a_{2\nu-1}$ to $a_{2\nu}$, $\nu = 1,2,\cdots,n$, and let

$$\begin{aligned} p_1(z) &= \text{branch of } \sqrt{\prod_1^{2n}(z-a_\nu)} \quad \text{reg. u.h.p.} \\ p_2(z) &= \text{branch of } \sqrt{\prod_1^{2n}(z-a_\nu)} \quad \text{reg. l.h.p.} \end{aligned} \tag{25}$$

Then

$$\lim_{y \to 0^+} \frac{p_1(x+iy)}{p_2(x-iy)} = -\operatorname{sgn} \lambda \operatorname{sgn}(r(x)-\lambda) , \tag{26}$$

so that if we define

$$Q_1(z) = -\operatorname{sgn} \lambda \, K_1(z) p_1(z), \quad \operatorname{Im} z > 0, \tag{27}$$

$$Q_2(z) = K_2(z) p_2(z), \quad \operatorname{Im} z < 0, \tag{28}$$

then it follows from (24) and (26) that

$$\lim_{y \to 0^+} \frac{Q_2(x-iy)}{Q_1(x+iy)} = r(x)-\lambda . \tag{29}$$

We may actually write

$$r(x)-\lambda = \frac{Q_2(x)}{Q_1(x)} \tag{29a}$$

except at the a_v.

We proceed to derive some estimates on Q_1 and Q_2. Near a zero $\underline{a}$, we have by (22)

$$|K_2(z)| \geq c \, |z-a|^{\frac{1}{2}}$$

and, by (23)

$$|K_1(z)| \leq c \, |z-a|^{-\frac{1}{2}} .$$

Hence, by (25),

$$|Q_2(z)| \geq c\,|z-a| \tag{30}$$

and

$$|Q_1(z)| \leq c\ , \tag{31}$$

near the zeros. Since K_1 and K_2 are bounded in their respective half planes, we have

$$|Q_2(z)| \leq c\,(1+|z|^n)\ , \qquad y<0, \tag{32}$$

and

$$|Q_1(z)| \leq c\,(1+|z|^n) \qquad y>0\ . \tag{33}$$

To bound Q_1 in the opposite direction, we recall that $r(x)$ is <u>bounded</u>, $\alpha \leq r(x) \leq \beta$.

$$|r(x)-\lambda| \leq c\ ,$$

and hence $|K_1(z)| \geq M > 0$ near the real axis. Thus

$$|Q_1(z)| \geq M|z-a|^{\frac{1}{2}} \tag{34}$$

near a zero <u>a</u>.

Returning now to (20), the result of the convolution lemma, we have

(35) $$\lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} \left| \frac{Q_2(x)}{Q_1(x)} F(x-i\varepsilon) - G(x+i\varepsilon) \right|^2 dx = 0.$$

Since, by (33), Q_1 is bounded in finite pieces of the real line, this implies, for any finite b,

(36) $$\lim_{\varepsilon \to 0^+} \int_{-b}^{b} |Q_2(x)F(x-i\varepsilon) - Q_1(x)G(x+i\varepsilon)|^2 dx = 0.$$

The first term of (36) is a function analytic in the lower half plane, and the second in the upper half plane. A version of the reflection principle is now required to prove that they are analytic continuations of one another.

Let Γ_1 and Γ_2 be semicircles defined as follows:

(37)
$$\Gamma_1 = (|z| = b, \ y \geq 0) \cup (|x| \leq b)$$
$$\Gamma_2 = (|z| = b, \ y \leq 0) \cup (|x| \leq b) .$$

Then

(38)
$$\frac{1}{2\pi i} \frac{1}{z^2-b^2} \int_{\Gamma_1} \frac{Q_1(\zeta)G(\zeta+i\varepsilon)(\zeta^2-b^2)}{\zeta-z} d\zeta = \begin{cases} Q_1(z)G(z+i\varepsilon) & \text{if } z \text{ is inside } \Gamma_1 \\ 0 & \text{if } z \text{ is inside } \Gamma_2 \end{cases}$$

and

(39)

$$\frac{1}{2\pi i}\,\frac{1}{z^2-b^2}\int_{\Gamma_2}\frac{Q_2(\zeta)F(\zeta-i\varepsilon)(\zeta^2-b^2)}{\zeta-z}\,d\zeta=\begin{cases}0 & \text{if } z \text{ is inside } \Gamma_1\\ Q_2(z)F(z-i\varepsilon) & \text{if } z \text{ is inside } \Gamma_2\,.\end{cases}$$

Here the integrals are both taken in the counterclockwise direction. Thus the sum $\Phi_\varepsilon(z)$ of the integrals in (38) and (39) represents

$$Q_1(z)G(z+i\varepsilon) \text{ in } (|z|<b)\cap(y>0) \quad \text{and}$$

$$Q_2(z)F(z-i\varepsilon) \text{ in } (|z|<b)\cap(y<0). \quad \text{But then}$$

(40)
$$\Phi_\varepsilon(z)=\frac{1}{2\pi i}\,\frac{1}{z^2-b^2}\left\{\int_{|\zeta|=b}\frac{\Phi_\varepsilon(\zeta)(\zeta^2-b^2)}{\zeta-z}d\zeta+\int_{-b}^{b}\frac{Q_1(\zeta)G(\zeta+i\varepsilon)-Q_2(\zeta)F(\zeta-i\varepsilon)}{\zeta-z}(\zeta^2-b^2)d\zeta\right\}.$$

Now, by (17) and (19),

$$|F(x-i\varepsilon)|\le\frac{c}{\varepsilon}$$

and
$$|G(x+i\varepsilon)|\le\frac{c}{\varepsilon^{\frac{1}{2}}}\,,$$

so that as $\varepsilon\to 0^+$,

$$\Phi_\varepsilon(\zeta)(\zeta^2-b^2)$$

tends boundedly to a limit at $\zeta=\pm b$; at all other points of $|\zeta|=b$, $\Phi_\varepsilon(\zeta)(\zeta^2-b^2)$ remains analytic as $\varepsilon\to 0^+$. Thus

the first integrand in (40) tends to a limit boundedly as $\varepsilon \to 0^+$. The second integral, for Im $z \neq 0$, is bounded by

$$\max \left| \frac{\zeta^2 - b^2}{\zeta - z} \right| \int_{-b}^{b} |Q_1(\zeta)G(\zeta+i\varepsilon) - Q_2(\zeta)F(\zeta-i\varepsilon)| d\zeta$$

and tends to zero as $\varepsilon \to 0^+$. Thus we see that

$$\lim_{\varepsilon \to 0^+} \Phi_\varepsilon(z) ,$$

which equals $Q_1(z)G(z)$ and $Q_2(z)F(z)$ in the open upper and lower halves of $|z| < b$ respectively, is represented by a Cauchy integral, and hence is an analytic function in $|z| < b$. Since b is arbitrary, we obtain that Q_1G and Q_2F are representations of one entire function $E(z)$.

As for estimates on $E(z)$, we have from (17) and (19), together with (32) and (33), that

$$|E(x+iy)| \leq c \frac{1+|z|^n}{|y|} \text{ if } |y| \leq 1$$

(41)

$$|E(x+iy)| \leq c \frac{1+|z|^n}{|y|^{\frac{1}{2}}} \text{ if } |y| > 1 .$$

We wish to prove E a polynomial. To do this, define the entire function

$$I(z) = \int_i^z E(\zeta)d\zeta \ .$$

For $0 < y \leq 1$, then, by integration from i to x+i and x+i to x+iy,

$$|I(x+iy)| \leq c\ |x|\left[1+|x+i|^n\right] + c\ |\ln|y||\left[1+|x+i|^n\right] \ .$$

If $1 < y$,

$$|I(x+iy)| \leq c\ |x|\left[1+|x+i|^n\right] + c\ |y|^{\frac{1}{2}}\left[1+|z|^n\right] \ .$$

If $y < 0$, we integrate first from i to -i and then apply the same bounds. Noting that $||x|+i| \leq |z| + 1$, we have, for all y,

$$|I(z)| \leq c_1 + c_2\ |z|(|z|+1)^n + c_3\left[|\ln|y||+|y|^{\frac{1}{2}}\right]\left[1+(|z|+1)^n\right] \ .$$

Then

$$I^{(m)}(0) = \frac{m!}{2\pi i} \int_{|z|=r} \frac{I(z)dz}{z^{m+1}} \ ,$$

and

$$|I^{(m)}(0)| < \frac{m!}{2\pi} \int_0^{2\pi} \frac{rd\theta}{r^{m+1}} \left\{ c_1 + c_4(1+(r+1)^n)(r+\ln r + |\ln|\sin\theta||+r^{\frac{1}{2}}) \right\} \ ,$$

and since

$$\int_0^{2\pi} |\ln|\sin\theta||\,d\theta < \infty ,$$

we have, by letting r tend to infinity, that $I^{(m)}(0) = 0$ for $m \geq n+1$. Hence $E(z)$ is a polynomial of degree not exceeding n. From the estimate (19), i.e.

$$|G(x+iy)| \leq \frac{c}{|y|} \qquad \text{as } y \to \infty ,$$

we see that $E(z)$ is actually of degree $\leq n-1$.

We have thus proved that $Q_2(z)F(z)$ is a polynomial of degree $(n-1)$. Hence the transform $F(z)$ of any solution of the integral equation (11) must have the form

$$F(z) = \int_0^{\infty} f(t)e^{-itz}\,dt = \frac{\sum_{\nu=0}^{n-1} c_\nu z^\nu}{Q_2(z)} .$$

It remains to prove that any $F(z)$ of this form is the transform of a function $f(t) \in B_1^2$, and that such an $f(t)$ must actually satisfy the integral equation.

In order to prove that $f(t) \in B_1^2$, we begin with the Parseval relation, for $\varepsilon > 0$,

(42) $$\int_{-\infty}^{\infty} |f(t)|^2 e^{-2\varepsilon|t|}\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{E(x-i\varepsilon)}{Q_2(x-i\varepsilon)}\right|^2 dx .$$

The latter integral is finite since $|Q_2|^2$ does not vanish off the real axis and behaves like $|z|^{2n}$ for large $|z|$, while $|E|^2$ behaves like $|z|^{2n-2}$. Hence

(43) $$\int_{-\infty}^{\infty} |f(t)|^2 e^{-2\varepsilon|t|}\,dt = \int_{-b}^{b} + \int_{|t|>b} |f(t)|^2 e^{-2\varepsilon|t|}\,dt < \infty .$$

If $b > \max(|a_o|, |a_{2n}|)$, then the integral over $|t| > b$ remains <u>bounded</u> as $\varepsilon \to 0$. On the other hand, E is bounded over a finite interval, and by (30), $|Q_2(z)| \geq c\,|z-a|$ near a zero. Hence

$$\frac{1}{2\pi}\int_{-b}^{b} \left|\frac{E(x-i\varepsilon)}{Q_2(x-i\varepsilon)}\right|^2 dx \leq M \int_{-b}^{b} \sum \frac{1}{|x-a_\nu+i\varepsilon|^2}\,dx < \frac{M_1}{\varepsilon} .$$

Thus

$$\int_{-\infty}^{\infty} |f(t)|^2 e^{-2\varepsilon|t|}\,dt < \frac{M_1}{\varepsilon} + M_2, \quad \text{and}$$

$$\int_{-1/\varepsilon}^{1/\varepsilon} |f(t)|^2\,dt \leq e^2 \int_{-1/\varepsilon}^{1/\varepsilon} |f(t)|^2 e^{-2\varepsilon|t|}\,dt < \frac{c_1}{\varepsilon} + c_2 .$$

We see that

$$\int_{-a}^{a} |f(t)|^2 dt < c\ (1+2a)\ ,$$

and hence

$$f(t)\ \varepsilon\ B_1^2\ .$$

To prove that any $f(t)$ of the form (14) must satisfy the integral equation (11), we define

$$h(t) = \int_{-\infty}^{\infty} \rho(t-s)f(s)ds - \lambda f(t) - g(t)\ , \tag{44}$$

where $g(t)$ is the function whose Fourier Transform for $y > 0$ is

$$G(z) = \frac{E(z)}{Q_1(z)}\ .$$

By the argument just concluded, $f(t)\ \varepsilon\ B_1^2$, and hence, by property d), $\int_{-\infty}^{\infty} \rho(t-s)f(s)ds\ \varepsilon\ B_1^2$. The same argument as just concluded for $f(t)$ shows that $g(t)$, with transform $E(z)/Q_1(z)$, is also $\varepsilon\ B_1^2$, so that $h(t)\ \varepsilon\ B_1^2$. Let

$$H(x,\varepsilon) = \int_{-\infty}^{\infty} h(t)e^{-\varepsilon|t|}e^{-ixt}dt\ . \tag{45}$$

In order to apply the proof of the second half of the convolution lemma, we need only to show that for arbitrary b_1 and b_2,

$$\lim_{\varepsilon \to 0^+} \int_{b_1}^{b_2} |H(x,\varepsilon)|\,dx = 0. \tag{46}$$

The convolution lemma applied to (44) gives us that

$$\lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} \left| \frac{Q_2(x)}{Q_1(x)} \frac{E(x-i\varepsilon)}{Q_2(x-i\varepsilon)} - \frac{E(x+i\varepsilon)}{Q_1(x+i\varepsilon)} - H(x,\varepsilon) \right|^2 dx = 0 \tag{47}$$

If (b_1, b_2) is a closed interval between zeros of $r(x)-\lambda$, and hence also of Q_1 and Q_2, then

$$\lim_{\varepsilon \to 0^+} \int_{b_1}^{b_2} \left| \frac{Q_2(x)}{Q_1(x)} \frac{E(x-i\varepsilon)}{Q_2(x-i\varepsilon)} - \frac{E(x+i\varepsilon)}{Q_1(x+i\varepsilon)} \right|^2 dx = 0,$$

and hence

$$\int_{b_1}^{b_2} |H(x,\varepsilon)|\,dx < c \left(\int_{b_1}^{b_2} |H(x,\varepsilon)|^2 dx \right)^{\frac{1}{2}} \to 0$$

as well.

Now to consider an interval $(a-\delta, a+\delta)$ containing a zero $\underline{a}$ of $r(x)-\lambda$. By the triangle inequality,

$$\int_{a-\delta}^{a+\delta} |H(x,\varepsilon)|\,dx \le o(1) + \int_{a-\delta}^{a+\delta} \left| [r(x)-\lambda] \frac{E(x-i\varepsilon)}{Q_2(x-i\varepsilon)} - \frac{E(x+i\varepsilon)}{Q_1(x+i\varepsilon)} \right| dx .$$

Now E is bounded, and by (34), $1/Q_1(x) \in L^1$, so that

$$\int_{a-\delta}^{a+\delta} \left| \frac{E(x+i\varepsilon)}{Q_1(x+i\varepsilon)} \right| dx = o(1) .$$

Furthermore,

$$\int_{a-\delta}^{a+\delta} |r(x)-\lambda| \frac{|E(x-i\varepsilon)|}{|Q_2(x-i\varepsilon)|} dx \le M \int_{a-\delta}^{a+\delta} \frac{|r(x)-\lambda|}{|Q_2(x-i\varepsilon)|} dx \le M_1 \int_{a-\delta}^{a+\delta} \left| \frac{K_2(x-i\varepsilon)}{p_2(x-i\varepsilon)} \right| dx$$

by (22) and (28). K_2, in turn, is bounded, and $p_2 \ge c\,|x-a|^{\frac{1}{2}}$.

Hence $\int_{a-\delta}^{a+\delta} |r(x)-\lambda| \left| \frac{E(x-i\varepsilon)}{Q_2(x-i\varepsilon)} \right| = o(1)$ also, and thus

$$\int_{a-\delta}^{a+\delta} |H(x,\varepsilon)|\,dx$$

can be made arbitrarily small by picking δ and ε small enough.

Thus $\lim_{\varepsilon \to 0^+} \int_{b_1}^{b_2} |H(x,\varepsilon)|\,dx = 0$ for arbitrary b_1 and b_2, and the theorem is proved.

Remarks: 1. Consider the equation

$$(48) \qquad \int_0^\infty \rho(t-s)f(s)ds = \lambda \int_0^\infty \rho_1(t-s)f(s)ds, \qquad t > 0 ,$$

where $\rho(1+|t|)^{\frac{1}{2}}$ and $\rho_1(1+|t|)^{\frac{1}{2}} \in A_1^2$ and $f \in B_1^2$. The previous analysis will hold for $r(x)-\lambda$ replaced by $r(x)-\lambda r_1(x)$, except that now $r(x)-\lambda r_1(x)$ tends to 0 at infinity, while it was bounded from both 0 and ∞ before. We now assume

$$\int_{-\infty}^{\infty} \frac{\log|r(x)-\lambda r_1(x)|\,dx}{1+x^2} > -\infty ,$$

so that the Poisson Integral (21) can be formed. Then the argument proceeds as before through the proof that $Q_2F = Q_1G = E$, where E is a polynomial which in the previous case was of degree $\leq n-1$. But now $Q_2 = K_2 P_2$ is not like z^n at ∞; P_2 is still like z^n, but K_2, the exponential of the Poisson Integral of $\frac{1}{2}\log|r(x)-\lambda r_1(x)|$, now tends to zero. If, for example, $r(x)-\lambda r_1(x) \sim c\,|x|^{-2v}$ at ∞, then $K_2 \sim c_1\,|x|^{-v}$, and the degree of E is $n-1-v$. Hence the number of solutions in B_1^2 is reduced; if $v > n-1$, then there are no solutions to (48) in B_1^2.

2. If the zeros of $r(x)-\lambda$ are not all simple, then from the expression

$$F(z) = \frac{E(z)}{Q_2(z)} ,$$

we cannot prove that $f \in B_1^2$, unless E vanishes at the zeros of Q_2 to an order within one of the order of the zeros of Q_2. This cuts down the number of free constants in E(z) so that the order of the eigenvalue will be reduced.

If the zeros of Q_2 are all of integer order, i.e., $\sim |z-a|^n$, then there are always solutions in B_1^2. The order of the eigenvalue is equal to the number of zeros regardless of their multiplicity. For if the order of a_v is k_v, then k_v-1 constraints on E are required to keep $f \in B_1^2$, and hence the number of free constants left in E is $\Sigma\, k_v - \Sigma\, (k_v-1) = v$.

3. If r(x) is a piecewise rational function, then in each of a number of intervals, $\log|r(x)-\lambda| = \Sigma\, c_v \log|x-a_v|$. The Poisson Integral of this requires the evaluation of

$$\int_{b_1}^{b_2} \frac{\log(\xi-a_i)\,d\xi}{\xi-z} ,$$

and while this is not expressible in closed form in terms of the common transcendental functions, it can be tabulated in terms of one complex parameter only. Thus problems of this kind are accessible to machine computation.