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BLACK NOTEBOOK II. Softcover
P 1 — 100
NOTES P 112 – 116
UPPSALA, SWEDEN
FALL 1944.

Hela Funktioner H.T. 44.

$|f(x+iy)| \le A e^{a|y|}$ A, a godtyckliga $< \infty$.

Klass A.

$\sup_{-\infty<x<\infty} |f(x+iy)| = e^{\mu(y)}$

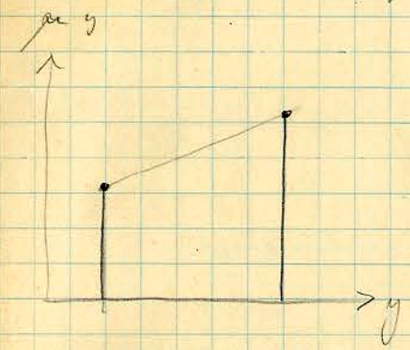
$\mu(y) \le a|y| + \log A$

$\sup. |f(x+iy) e^{i\lambda(x+iy)}| = e^{\mu(y) - \lambda y}$

$y_1 < y_2 < y_3$; $\mu(y_1) - \lambda y_1 = \mu(y_3) - \lambda y_3$

$\lambda(y_3 - y_1) = \mu(y_3) - \mu(y_1)$

$\lambda = \dfrac{\mu(y_3) - \mu(y_1)}{y_3 - y_1}$



$\mu(y) - \lambda y = C$ $y = y_1 \text{ o. } y_3$

inl. max. pr. $\mu(y) - \lambda y \le C$ $y_1 < y < y_3$

$\mu(y) \le \lambda y + C$ ( )

$\mu'(y)$ monoton funktion

$\lim_{y=\infty} \mu(y) = -a$ $\qquad e^{iaz}$

$\lim_{y=-\infty} \mu(y) = -b$

$-a \geq -b$

$a \leq b$

$a = b = \lambda$ $\qquad \mu(y) = -\lambda$

$\mu'(y) = -\lambda$

$\mu(y) = -2y + c$

$\sup_{R(z)=y} |f(z) e^{-i\lambda z - c}| = e^{\mu(y) + 2y - c} \equiv 1$

$f(x) = K \cdot e^{i\lambda x}$; $|K| = e^{c}$

(a, b) f.s. spektralintervall

$L^p(-\infty, \infty)$; $\left\{ \int_{-\infty}^{\infty} |f(x)|^p dx \right\}^{\frac{1}{p}} = \|f\|_p$ $\qquad p \geq 1$

$L^\infty = C$ $\qquad$ $f(x)$ likformig + kont. o. begränsad

$\|f\|_\infty = \sup_{-\infty < x < \infty} |f(x)|$

Sats. I Mängden A består av kontinuerliga L^p, $p \geq 1$ och C_0.

1) $\varphi \in C$, $\varepsilon > 0$; $f \in A$; $\sup_x |\varphi(x) - f(x)| < \varepsilon$

$\omega(\delta) = \sup_{\substack{-\infty < x < \infty \\ |h| \leq \delta}} |\varphi(x+h) - \varphi(x)|$; $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$, $\omega \downarrow 0$ $\delta \to 0$

1. $f(x+iy) = O\left(\frac{e^{|y|}}{x^2+y^2}\right)$
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\Big\} \leq C \left(\frac{\sin \frac{x}{2}}{x}\right)^2$

$\varphi_\mu(z) = \int_{-\infty}^{\infty} \varphi(\xi)\, f(\mu z - \mu\xi)\, \mu\, d\xi \in A$ $\qquad \mu > 0$

$f(\mu x + \mu i y - \mu\xi) = O\left\{\frac{e^{\mu|y|}}{y^2 + (x-\xi)^2}\right\}$

$\varphi_\mu(x) = \int_{-\infty}^{\infty} \varphi(\xi)\, f(\mu x - \mu\xi)\, \mu\, d\xi = \int_{-\infty}^{\infty} \varphi(x - \tfrac{\eta}{\mu}) f(\eta)\, d\eta$ $\qquad \left[\begin{array}{l} \mu x - \mu\xi = \eta \\ \xi = x - \frac{\eta}{\mu} \end{array}\right]$

$\varphi_\mu(x) - \varphi(x) = \int_{-\infty}^{\infty} (\varphi(x - \tfrac{\eta}{\mu}) - \varphi(x))\, f(\eta)\, d\eta$

$|\varphi_\mu(x) - \varphi(x)| \leq \int_{-\infty}^{\infty} \omega(\tfrac{\eta}{\mu}) |f(\eta)|\, d\eta = \omega(\tfrac{a}{\mu}) \int_{-a}^{a} |f(\eta)|\, d\eta + 2M \int_{|\eta| > a} |f(\eta)|\, d\eta$

$< \frac{\varepsilon}{2}$ $\qquad < \frac{\varepsilon}{2}$

1. $\varphi \in L$; $\int_{-\infty}^{\infty} |\varphi_n(x) - \varphi(x)|\,dx < \varepsilon$

$\int_{-\infty}^{\infty} |\varphi(x+\delta) - \varphi(x)|\,dx = \omega(\delta)$; $\omega(\delta) \to 0$, $\omega(\delta) \le 2\int_{-\infty}^{\infty} |\varphi|\,dx$

$\int_{-\infty}^{\infty} |\varphi_n(x) - \varphi(x)|\,dx \le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\varphi(x+\eta_n) - \varphi(x)|\,|f(\eta)|\,d\eta\,dx =$

$= \int_{-\infty}^{\infty} \omega(\eta_n)\,|f(\eta)|\,d\eta < \varepsilon.$

2. $\varphi \in L^p$; $1 < p < \infty$

$\int_{-\infty}^{\infty} |\varphi_n(x) - \varphi(x)|^p\,dx \le \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} |\varphi(x+\eta_n) - \varphi(x)|\,|f(\eta)|\,d\eta\right\}^p dx \le$

$\le \int_{-\infty}^{\infty} |\varphi(x-\eta_n) - \varphi(x)|$

Höldens olikhet: $|\varphi_n(x) - \varphi(x)| \le$

$\le \int_{-\infty}^{\infty} |\varphi(x-\eta_n) - \varphi(x)|^p |f(\eta)|\,d\eta\Big\}^{\frac{1}{p}} \Big\{\int_{-\infty}^{\infty} |f(\eta)|\,d\eta\Big\}^{\frac{1}{q}}$

$\alpha = \Delta f^{\frac{1}{p}}$
$\beta = f^{\frac{1}{q}}$

$|\varphi_n(x) - \varphi(x)|^p \le C \cdot \int_{-\infty}^{\infty} |\varphi(x-\eta_n) - \varphi(x)|^p |f(\eta)|\,d\eta$

$\int |\varphi_n(x) - \varphi(x)|^p dx \le C \int_{-\infty}^{\infty} \omega^p(\eta_n)\,|f(\eta)|\,d\eta$

Sats. II Om $f(x)$ har spektralintervallet (a,b) så har $f(kx)$ (ka, kb), $e^{i\lambda x} f(kx)$ $(ka+\lambda, kb+\lambda)$

$\mu(y)$
$\mu(ky)$
$\mu(ky) - \lambda y$

"Svag konvergens" av kontinuerliga funktioner på reella axeln:

$\varphi_n(x)$ konvergerar svagt mot $\varphi(x)$ om 1) konvergerar likformigt på varje ändligt intervall 2) om $\lim_{n\to\infty} M(\varphi_n) = M(\varphi)$

$\lambda_n \to \lambda$ $e^{i\lambda_n x}$ konv. svagt mot $e^{i\lambda x}$

$|e^{i\lambda_n x} - e^{i\lambda x}| = |e^{i(\lambda-\lambda_n)x} - 1| = |e^{i\frac{\lambda-\lambda_n}{2}x - i\frac{\lambda-\lambda_n}{2}x}| =$

$= 2\,|\sin \underbrace{\tfrac{\lambda-\lambda_n}{2}}_{\to 0} x|$

$\varphi(x) = 0$
$\varphi_n(x) = e^{-(x-n)^2}$

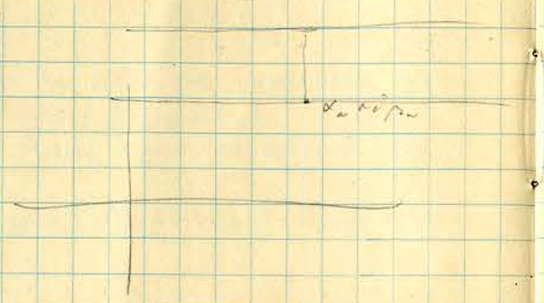
Sats III Anta 1) $f(x+iy)$ holomorf för $y > y_0$ (och $\neq 0$) samt 2) $|f(x+iy)| < Ae^{ay}$ $y > y_0$

3) $\lim_{y\to\infty} \mu'(y) = -\lambda$

Då kan alltid en följd punkter a_n utväljas i halvplanet så att funktionerna $\frac{f(x+a_n)}{f(a_n)}$ konvergerar svagt mot $e^{i\lambda x}$ på reella axeln.

$a_n = \alpha_n + i\beta_n \quad ; \quad \beta_n \to \infty$

$$\sup_{J(z)=y} |f(z+\alpha_n+i\beta_n)| = \sup_{J(z)=y+\beta_n} |f(z)| = e^{\mu(y+\beta_n)}$$



$\mu'(y) \to -\lambda$

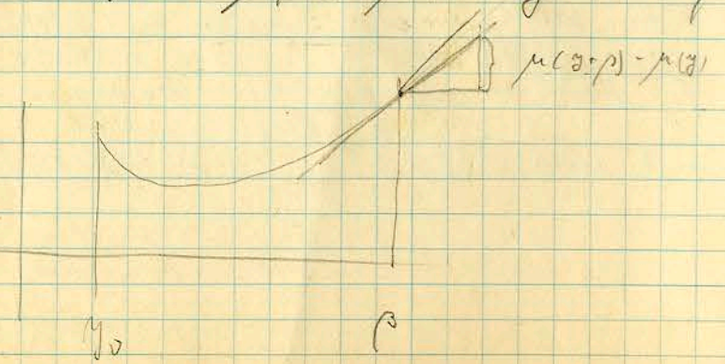
$\frac{d}{dy}(\mu(y)+\lambda y) \to 0 \qquad y \to \infty$

$$|f(\alpha_n+i\beta_n)| \geq \frac{\beta_n}{\beta_n+1} \cdot e^{\mu(\beta_n)}$$

$$\sup_{J(z)=y} \left| \frac{f(z+\alpha_n+i\beta_n)}{f(\alpha_n+i\beta_n)} \underbrace{e^{-i\lambda z}}_{e^{\lambda y}} \right| = \frac{e^{\mu(\beta_n)}}{|f(\alpha_n+i\beta_n)|} \cdot e^{\mu(y+\beta_n)-\mu(\beta_n)+\lambda y}$$

$$\leq \frac{\beta_n+1}{\beta_n} \cdot e^{\mu(y+\beta_n)-\mu(\beta_n)+\lambda y}$$

$$\int_\beta^{\beta+y} \frac{d}{dy}(\mu(y)+\lambda y) = \mu(\beta+y)-\mu(\beta)+\lambda y \to 0 \quad \beta \to \infty$$



$$g_n(z) = e^{-i\lambda z} \frac{f(z+\alpha_n+i\beta_n)}{f(\alpha_n+i\beta_n)}$$

$|g_n(x+iy)| \leq 2 \qquad -\rho_n \leq y \leq \rho_n \quad ; \quad \rho_n \to \infty.$

$$\frac{\beta_{n+1}}{\beta_n} e^{\mu(y+\beta_n)-\mu(\beta_n)+\lambda y} \leq 2 \qquad -\rho_n \leq y \leq \rho_n$$

$g_n(0) = 1 \quad ; \quad |g_n(x)| \leq \frac{\beta_{n+1}}{\beta_n} \quad ; \quad |g_n'(x)| \leq \frac{2}{\rho_n}$

$|g_n(x) - 1| \leq \frac{2|x|}{\rho_n}$

$g_n(x)$ konvergerer svagt mot 1 på nella axelen

$\therefore \; g_n(x)e^{i\lambda x}$ " " $e^{i\lambda x}$ " " ".

$\sin x = \frac{e^{ix}-e^{-ix}}{2i} \quad ; \quad \sin(x+iy) = \frac{e^{ix-y}-e^{-ix+y}}{2i}$

$$\frac{\sin(x+i\beta)}{\sin(i\beta)} = \frac{-e^{ix-\beta}+e^{-ix+\beta}}{2i} \cdot \frac{2i}{-e^{-\beta}+e^{+\beta}} = \qquad \frac{e^{y}+e^{-y}}{2}$$

$$= \frac{e^{-ix}e^{\beta}(1-e^{+2ix-2\beta})}{e^{\beta}(1-e^{-2\beta})} = e^{-ix} \cdot \frac{1-e^{-2\beta+2ix}}{1-e^{-2\beta}}.$$

$$= e^{-ix} + \delta_n(x); \quad |\delta_n(x)| \leq \frac{2e^{-2\beta}}{1-e^{-2\beta}} < 3e^{-2\beta}$$

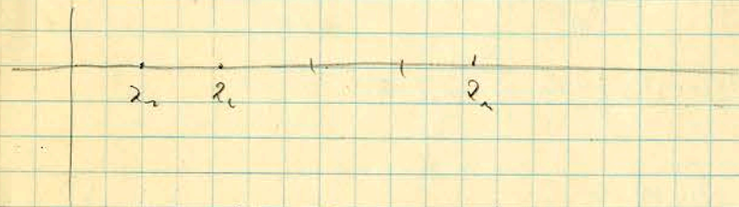
$$P(\varphi) = \sum_1^n A_n P(x, e^{i\lambda_n x}) = \sum_1^n A_n P(\tau_n) e^{i\lambda_n x}$$

$$Q(\varphi) = \sum_1^{n'} A_n' Q(x, e^{i\lambda_n x}) = \sum_1^{n'} A_n' Q(\tau_n) e^{i\lambda_n x}$$

$$QP(\varphi) = \sum A_n Q(\tau_n) P(\tau_n) e^{i\lambda_n x}$$

$$QP = PQ = R$$

$$P(\lambda_1) = \frac{1}{A_1} \; ; \; P(\lambda_i) = 0 \quad i = 2, 3 \dots n$$



$$P(\tau) = P_1(\tau)\, P_2(\tau) \cdots P_{n-1}(\tau)$$

$P_1(\lambda_1) \neq 0$, $P_1(\lambda_2) = 0$; $P_2(\lambda_2) \neq 0$, $P_2(\lambda_3) = 0$; $P_{n-1}(\lambda_{n-1}) \neq 0$, $P_{n-1}(\lambda_n) = 0$

$$1 - e^{ia(\tau - \lambda_2)} = 0 \; ; \qquad Q(\tau - \lambda_2) = 2m\pi$$

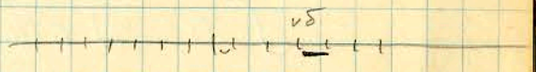
$$\tau = \lambda_2 + \frac{2m\pi}{a}$$

Om $\varphi(x) = \sum_1^n A_\nu e^{i\lambda_\nu x}$ $A_\nu \neq 0$

innehåller T_φ funktionen $e^{i\lambda_\nu x}$ men icke funktionen $e^{i\alpha x}$ där $\alpha \neq \lambda_\nu$

Om $\int_{-\infty}^{\infty} |f(\tau)|\, d\tau < \infty$ så innehåller T_φ funktionen

$$\int_{-\infty}^{\infty} \varphi(x+\tau) f(\tau)\, d\tau$$



$$\eta = \sup \left| \sum_{-\infty}^{\infty} \varphi(x+\nu\delta) \int_{\nu\delta}^{(\nu+1)\delta} f(\tau)\, d\tau - \int_{-\infty}^{\infty} \varphi(x+\tau) f(\tau)\, d\tau \right| \leq$$

$$\leq \sum_{\nu=-\infty}^{\infty} \int_{\nu\delta}^{(\nu+1)\delta} (\varphi(x+\nu\delta) - \varphi(x+\tau)) f(\tau)\, d\tau$$

$$\eta \leq \omega(\delta) \int_{-\infty}^{\infty} |f(\tau)|\, d\tau < \frac{\varepsilon}{2} \quad \text{om} \quad \delta < \delta(\varepsilon)$$

$$\left| \sum_{\nu=-N}^{\nu=+N} \varphi(x+\nu\delta) \int_{\nu\delta}^{(\nu+1)\delta} f(\tau)\, d\tau - \psi(x) \right| \leq \eta + \sum_{-\infty}^{-(N+1)\delta} + \sum_{(N+1)}^{\infty}$$

Enligt sats I finns en hel funktion $\varphi(x) \in A$ sådan att $|\varphi(x) - \psi(x)| \leq \varepsilon$ på $-\infty < x < \infty$

$$\psi(x) = \varphi_H(x) = \int_{-\infty}^{\infty} \varphi(\xi)\, f(Hx - H\xi)\, H\, d\xi$$

$$\psi(x) = \int_{-\infty}^{\infty} \varphi(\xi)\, f(x-\xi)\, d\xi \qquad ; \quad f(Hx)H \;\text{-}\!\!/\; f(x)$$

$$f(x+iy) = O\left(\frac{e^{H|y|}}{x^2+y^2}\right)$$

$$\psi(x+\alpha+i\beta) =$$

$$= \int_{-\infty}^{\infty} \varphi(\xi)\, f(x+\alpha+i\beta-\xi)\, d\xi = \qquad x-\xi = \eta, \quad \xi = x-\eta$$

$$= \int_{-\infty}^{\infty} \varphi(x-\eta)\, f(\eta+\alpha+i\beta)\, d\eta \subset T_\varphi$$

T_φ sluten relative vag konvergens: Om

$\varphi_n \subset T_\varphi$ och $\varphi_n \xrightarrow[\text{vagt}]{} \psi$ så $\psi \subset T_\varphi$

$\varepsilon > 0 \qquad |\varphi_n(x) - \psi(x)| \le \varepsilon \qquad$ för $-\frac{1}{\varepsilon} \le x \le \frac{1}{\varepsilon}$

$M(\varphi_n) \le M(\psi) + \varepsilon \qquad n > n_\varepsilon$

$|\varphi_n - P| \le \varepsilon \qquad$ för $\quad -\frac{1}{\varepsilon} \le x \le \frac{1}{\varepsilon}$

$M(P) \le M(\varphi_n) + \varepsilon \le M(\psi) + 2\varepsilon$

$$\begin{cases} |\psi - P| \le 2\varepsilon & \text{för } -\frac{1}{2\varepsilon} \le x \le \frac{1}{2\varepsilon} \\ M(P) \le M(\psi) + 2\varepsilon \end{cases}$$

$$\frac{\psi(x+\alpha+i\beta)}{\psi(x+i\beta)} \subset T_\varphi$$

$\psi(x)$ spektralintervall (a,b)

e^{+iax} och $e^{+ibx} \subset T_\varphi$

$$\varphi_H(x) = \int_{-\infty}^{\infty} \varphi(x - \eta/H)\, f(\eta)\, d\eta \xrightarrow[\text{likf.}]{} \varphi(x) \quad \text{när } H \to \infty$$

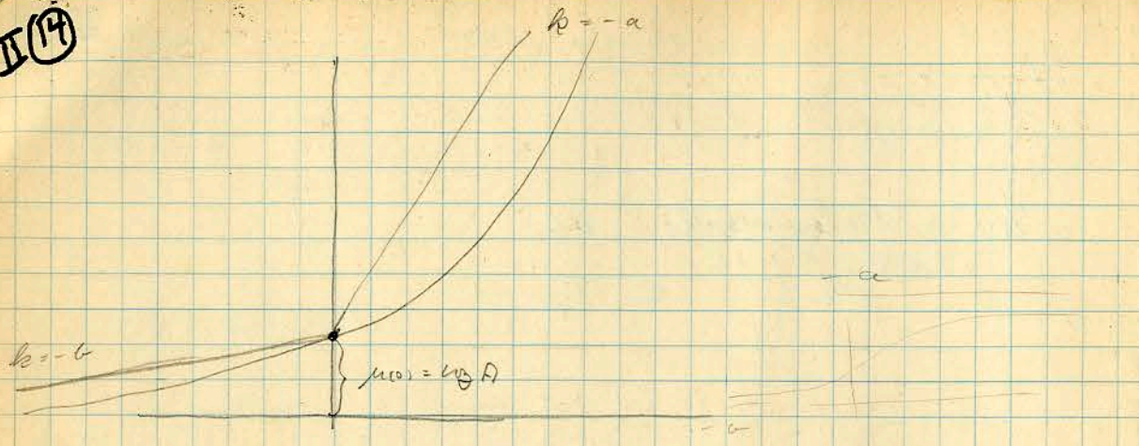
För varje H gäller att φ_H:s spektralintervall består av en punkt λ

$\mu_H(y) = -\lambda y + c \qquad \therefore \quad \varphi_H(x) = A(H)\, e^{i\lambda x} \xrightarrow[\text{likf.}]{} \varphi(x)$

$A(H) \to A \qquad$ när $H \to \infty$

$A \neq 0.$

Def. Mängden av tal λ för vilka $e^{i\lambda x} \subset T_\varphi$ bildar φ:s spektrum



$\lim_{y\to 0} \mu'(y) = -a$ $\qquad$ $\mu'(y) \le -a$

$\lim_{y\to -\infty} \mu'(y) = -b$ $\qquad$ $\mu(y) = \mu(0) + \int_0^y \mu'(y)\,dy \le$

$\mu(0) - ay \qquad y > 0$

$-\mu'(y) \le b$

$\mu(y) \le \mu(0) + by$

$$\mu(y) - \mu(0) = \int_0^y \mu'(y)\,dy = \int_y^0 \underbrace{-\mu'(y)}_{\le b}\,dy \le$$

$$\mu(y) \le \begin{cases} \mu(0) - ay & y \ge 0 \\ \mu(0) - by & y \le 0 \end{cases}$$

Lemma: Om $f \subset A$ ~~samt $|f(x)| \le 1$~~ och har spektralintervallet (a, b) samt $|f(x)| \le 1$ för $-\infty < x < \infty$ så gäller

$$|f(x+iy)| \le \begin{cases} e^{-ay} & y \ge 0 \\ e^{-by} & y \le 0 \end{cases}$$

Sats. V Spektrum för en hel funktion $\varphi(x) \subset A$ är innesluttet i spektralintervallet (a, b). Punkterna a och b tillhöra spektret.

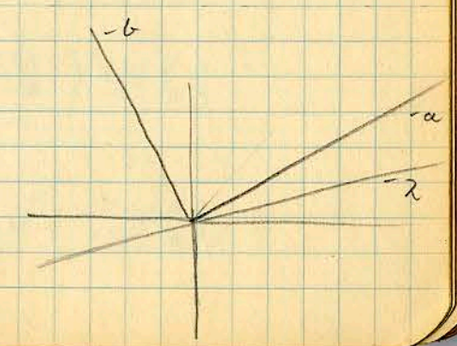
$$P(x) = \sum_1^n c_\nu \varphi(x + \tau_\nu)$$

$$|P(x)| \le 1 \; ; \; \sup |P(x+iy)| \le \sum_1^n |c_\nu| e^{\mu_\varphi(y)}$$

$$|P(x+iy)| \le \begin{cases} e^{-ay} & y \ge 0 \\ e^{-by} & y \le 0 \end{cases}$$

$P_n(x) \xrightarrow[\text{sorgt}]{} e^{i\lambda x}$ $\qquad$ konvergens likf.

i om varje cirkel $|z| \le R$



Lemma: Om $f(x) \subset H$ har spektralintervallet $(-a, a)$ samt $f(x) \leq M$ så gäller

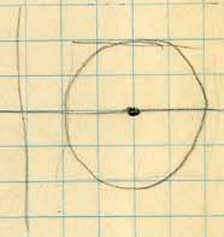
$$|f'(x)| \leq e \cdot a \cdot M \qquad (e \text{ bran eviktor med } 1)$$

$$\mu(y) \leq \log M + a|y|$$

$$|f(x+iy)| \leq M \cdot e^{a|y|}$$

$$f'(x) = \frac{1}{2\pi i}\int_C \frac{f(t)}{(t-x)^2}\,dt$$

$$|f'(x)| \leq M \cdot \frac{e^{a\rho}}{\rho} = eMa, \quad \rho = \frac{1}{a}$$



$$b - a = l$$

$$\left|\frac{d\left(e^{-i\frac{a+b}{2}x} f(x)\right)}{dx}\right| \leq \frac{e \cdot l}{2} M$$

$$\left|f'(x) - i\frac{a+b}{2} f(x)\right| \leq e \cdot \frac{b-a}{2} M.$$

Specialfall av Fouriers integral formel.

Om
$$f(x+iy) = \begin{cases} O\left(\dfrac{e^{-ay}}{x^2+y^2}\right) & y \geq 0 \\ O\,\dfrac{e^{-by}}{x^2+y^2} & y \leq 0 \end{cases}$$

så gäller att

$$\int_{-\infty}^{\infty} f(x) e^{-itx}\,dx = h(t)$$

där $h(t)$ kontinuerlig funktion $= 0$ för $t \leq a$ och $t \geq b$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(t) e^{itx}\,dt$$

$$e^{-it(x+iy)}$$

$$|f(z) e^{-itz}| \leq \frac{K}{|z|^2} \cdot e^{-by+ty} = \frac{K}{|z^2|} e^{y(t-b)} < \frac{K}{z^2} \qquad t \geq b,\ y \leq 0$$

$$\frac{K}{|z^2|} e^{-ay+ty} = \frac{K}{|z|^2} e^{-y(a-t)} \leq \frac{K}{z^2} \qquad t \leq a,\ y \geq 0$$

$$\frac{1}{2\pi}\int_a^b h(t) e^{itx}\,dt = \frac{1}{2\pi}\int_{-l}^{l} e^{itx} \int_{-\infty}^{\infty} f(\xi) e^{-it\xi}\,d\xi\,dt$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin l(x-\xi)}{x-\xi} f(\xi)\,d\xi =$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin l\eta}{\eta} f(x-\eta)\,d\eta$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin\eta}{\eta}\, f\left(x-\frac{\eta}{\ell}\right) d\eta$$

$$J_\ell(x) = \frac{1}{2\pi}\int_{-\ell}^{\ell} h(t)\, e^{itx}\, dt = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin\ell\eta}{\eta} f(x-\eta)\, d\eta =$$

$$= \frac{1}{\pi}\int_0^{\infty} \frac{\sin\ell\eta}{\eta}\, f(x+\eta)+f(x-\eta)\, d\eta$$

$$= \frac{1}{\pi}\int_0^{1} \frac{\sin\ell\eta}{\eta} 2f(x)\, d\eta + \frac{1}{\pi}\int_0^{1} \frac{\sin\ell\eta}{\eta} \{f(x+\eta)+f(x-\eta)-2f(x)\}\, d\eta +$$

$$+ \frac{1}{\pi}\int_1^{\infty} \frac{\sin\ell\eta}{\eta} (f(x+\eta)+f(x-\eta))\, d\eta$$

$$= f(x)\cdot\frac{2}{\pi}\int_0^{\ell} \frac{\sin\alpha}{\alpha}\, d\alpha + \frac{1}{\pi}\int_0^{\infty} \sin\ell\eta\, g(\eta)\, d\eta$$

$$g(\eta) = \begin{cases} \dfrac{f(x+\eta)+f(x-\eta)-2f(x)}{\eta} & 0\le\eta\le 1 \\ \dfrac{f(x+\eta)+f(x-\eta)}{\eta} & \eta\ge 1. \end{cases}$$

$g(\eta)$ är begränsad variation

$$\frac{1}{\pi}\int_a^{\infty} |g(\eta)|\, d\eta < \varepsilon$$

$$\frac{1}{\pi}\int_0^{a} \sin\ell\eta\, g(\eta)\, d\eta = \frac{1}{\pi}\int_0^{a} -\frac{\cos\ell\eta}{\ell}\, g(\eta) + \frac{1}{\pi\ell}\int_0^{a} \cos\ell\eta\, d\, g(\eta)$$

$$\le \frac{|g(a)|+|g(0)|}{\pi\ell} + \frac{1}{\pi\ell}\int_0^{a} |d\, g(\eta)|$$

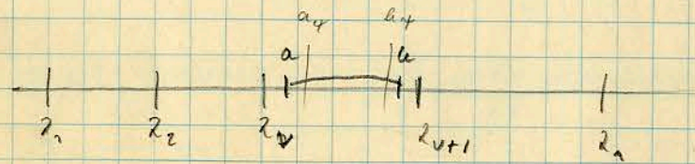
Lemma: Om $f(x)$ har speltalsintervallet (a,b) samt

$$|f(x+iy)| \le \frac{K}{|x|^2}\begin{cases} e^{-ay} & y>0 \\ e^{-by} & y\le 0 \end{cases}$$

så kan b_n alltid bestämmas så $\int_{-\infty}^{\infty} |f(x)-b_n(x)|\, dx < \varepsilon$ under det att f_n är speltalsintervall innehållet i (a,b).

Sats VI Om $\varphi(x)$ är en begr. o. likformigt kontinuerlig funktion vars spektrum endast innehåller ett ändligt antal punkter $\lambda_1, \lambda_2 \ldots \lambda_n$ så är

$$\varphi(x) \equiv \sum_1^n A_\nu e^{i\lambda_\nu x} \qquad A_\nu \neq 0$$



Hjälpfunktion $\quad |f(x+iy)| \leq \frac{K}{x^{1+\nu}} \begin{cases} e^{-ay} & y \geq 0 \\ e^{-by} & y \leq 0 \end{cases}$

$$\psi(x) = \int_{-\infty}^{\infty} \varphi(x-\xi) f(x)\,dx = \int_{-\infty}^{\infty} \varphi(\xi) f(x-\xi)\,d\xi$$

$$|\psi(x+iy)| \leq K' \begin{cases} e^{-ay} & y \geq 0 \\ e^{-by} & y \leq 0 \end{cases}$$

$\lim_{y\to+\infty} \mu_\psi'(y) = -a_\psi$

$\lim_{y\to-\infty} \mu_\psi'(y) = -b_\psi$

$$a \leq a_\psi \leq b_\psi \leq b$$

$e^{ia_\psi x}$ och $e^{ib_\psi x} \subset T_\varphi$ vilket strider mot förutsättningen.

$\therefore \psi \equiv 0$

$a = \lambda_\nu$; $b = \lambda_{\nu+1}$? $\qquad a_\psi = a = \lambda_\nu$; $b_\psi = b = \lambda_{\nu+1}$

$$\psi(x) = \int_{-\infty}^{\infty} \varphi(x-\xi) f(\xi)\,d\xi \qquad \text{Anlag } \psi \not\equiv 0$$

$\max_{\infty < x < \infty} |\psi(x)| = d > 0 \quad ; \quad |\psi(x_0)| > \frac{2d}{3}$

$f_1(x)$ med sp. intervall (a,b) innesluter med min indpunkter i $(\lambda_\nu, \lambda_{\nu+1})$

$$\int_{-\infty}^{\infty} |f_1(\xi) - f(\xi)|\,dx < \varepsilon$$

$$\psi_1(x) = \int_{-\infty}^{\infty} \varphi(x-\xi) f_1(\xi)\,d\xi \equiv 0$$

$$|\psi(x_0) - \psi_1(x_0)| \leq \left| \int_{-\infty}^{\infty} \varphi(x-\xi)(f(\xi) - f_1(\xi)) \right| \leq M(\varphi) \cdot \int |f - f_1|\,d\xi \leq \varepsilon M(\varphi)$$

$\lambda_\nu + \lambda_{\nu+1} = 0 \qquad \lambda_\nu$

$f_1(x) = f(x(1-\varepsilon)) \qquad \{(1-\delta)\lambda_\nu, (1-\delta)\lambda_{\nu+1}\}$

$$\int_{-\infty}^{\infty} |f(x) - f(x(1-\delta))|\,dx$$

3/10. $\Lambda_\varphi = \{\lambda_1, \lambda_2, \dots \lambda_n\}$ $\quad \varphi \equiv \sum^n A_\nu e^{i\lambda_\nu x}$

$P(D) = \prod^n (D - i\lambda_\nu)$;

$f(x) = \frac{1}{2\pi} \int h(t) e^{itx}\, dt$

1. $h(t)$ alg. kont. och diff.bara på $(-\infty, \omega)$ samt $=0$ utanför ändligt intervall
2. $h'(t)$ av begr. variation.

$h_1 + h_2 \quad h_1 h_2 \quad ; \quad h_1 h_2' + h_2 h_1'$

$h(t)$, $h(\frac{t}{\mu})$ $\quad \mu > 0$

$$f(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(t)\, e^{itz}\, dt = \frac{-1}{2\pi i z} \int_{-\infty}^{\infty} e^{itz} h'(t)\, dt =$$

$$= -\frac{1}{2\pi z^2} \int_{-\infty}^{\infty} e^{itz}\, d\, h'(t)$$

$h'(t) = 0$ utanför (a, b) .

$$|f(z)| \le \frac{\int |d h'(t)|}{2\pi |z|^2} \begin{cases} e^{-ay} & y \ge 0 \\ e^{-by} & y \le 0 \end{cases}$$

$$\frac{1}{2\pi} \int h(\tfrac{t}{\mu})\, e^{itx}\, dt = \mu \cdot \frac{1}{2\pi} \int h(s)\, e^{-is\mu x}\, ds = \mu f(\mu x)$$

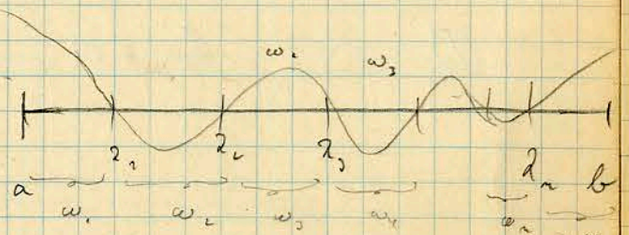
$\frac{t}{\mu} = s$

$D^\nu f(x) = \frac{1}{2\pi} \int (it)^\nu h(t) e^{itx}\, dt$ $\quad$ samma hölds

$P(D) f = \frac{1}{2\pi} \int P(it)\, h(t)\, e^{itx}\, dt$

$P(D) = \prod^n (D - i\lambda_\nu)$

$P(it) = (i)^n \prod^n (t - \lambda_\nu)$



$$P(D) f = \sum_{\nu=1}^{n+1} \frac{1}{2\pi} \int_{\omega_\nu} P(it)\, h(t)\, e^{itx}\, dt = \sum_1^{n+1} F_\nu(x)$$

$F_\nu(x) = \frac{1}{2\pi} \int_{\omega_\nu} P(it)\, h(t)\, e^{itx}\, dt$

1. $P(it)\, h(t)$ alg. kont., o. kont. på (ω, ∞) samt $=0$ utanför ω_ν
2. derivatan av begr. variation

$\therefore \int_{-\infty}^{\infty} \varphi(\xi)\, F_\nu(x - \xi)\, d\xi \equiv 0$

$\therefore \varphi_\mu(x) = \int_{-\infty}^{\infty} \varphi(\xi)\, f(\mu x - \mu\xi)\, \mu\, d\xi$

$P(D)\, \varphi_\mu = \int_{-\infty}^{\infty} \varphi(\xi)\, P(D_x)\, f(\mu x - \mu\xi)\, \mu\, d\xi \equiv 0$

Sats VII Om φ begr. o. likf. kont $(-\infty < t < \infty)$ samt spektrum ej har någon ändl. kopningspunkt ~~sigatter~~

$(\Lambda_\varphi = \{\lambda_1, \lambda_2, \dots\}$ $|\lambda_1| \le |\lambda_2| \le \dots \to \infty)$

så ~~kan $\varphi(x)$~~ är $\varphi(x)$ uniformt gränsvärde till polynom $\sum_1^n A_{\nu,n} e^{i\lambda_\nu t}$.

$\Psi_{\varphi_{\mu}}$ $\Delta_{\varphi_{\mu}}$ har ändl. spektrum

---

E vektorielt (komplext) rum des element P, Q, R,

$\underline{a}P + \underline{b}Q \subset E$

1) $\|P+Q\| \le \|P\| + \|Q\|$

2) $\|\underline{a}P\| = |\underline{a}| \cdot \|P\|$

$T_\tau P = P_\tau$; $\|T_\tau P\| = \|P\|$

$T_{\tau_1} T_{\tau_2} = T_{\tau_2} T_{\tau_1} = T_{\tau_1+\tau_2}$ isometrisk, linjär, abelsk grupp

$\|T_\varepsilon P - P\| \to 0$ $\varepsilon \to 0$ Elementen kontinuerliga på gruppen om

Rotation ett element Q som uppgyller relationen

$T_\tau Q = C(\tau) Q$; $|C(\tau)| = 1$.

$T_{\tau_1} T_{\tau_2} Q = T_{\tau_1} C(\tau_2) Q = C(\tau_2) T_{\tau_1} Q = C(\tau_2) C(\tau_1) Q = T_{\tau_1+\tau_2}(Q) =$

$C(\tau_1+\tau_2) Q$ $\because$ $C(\tau_1+\tau_2) = C(\tau_1) C(\tau_2)$

$C(0) = 1$; $|C(\tau)| = 1$ $|C(\tau+\varepsilon) - C(\tau)| \to 0$ $\varepsilon \to 0$

$C(\tau) = e^{i\psi(\tau)}$; $\psi(0) = 0$; $\psi(\tau)$ kont.

$\psi(\tau_1+\tau_2) = \psi(\tau_1) + \psi(\tau_2)$ $\because$ $\psi(\tau) = \lambda \tau$

λ reell kont.

$Q_\tau = e^{i\lambda\tau} Q$ rotation med frekvensen λ

tal: $A(P)$ linjär funktional i E ; $A(P_\tau) \equiv 0$ $P = O$

$|A(P)| \le \|P\|$

$A(P_\tau) = P(\tau)$

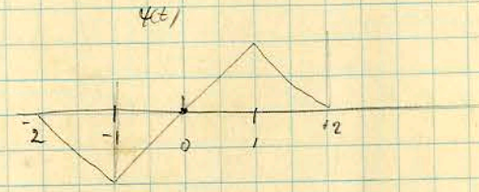
$P^{(n)}$ konvergerar svagt om $P^{(n)}(\tau)$ konvergerar svagt

$H_{\substack{\text{svaga}\\ \text{konv.}}}\{\sum c_\nu P_{\tau_\nu}\} \subset$ $T_a\{\int P_t \, d\mu(t)\} / P$

A

Sbs. IX Om $f(x) \subset A$ och har spektralintervallet $(-1,1)$ så gäller en utvecklingen

$$f'(x) = \sum a_\nu f\left(x - \frac{\pi}{2}(2\nu+1)\right)$$



$e^{\frac{2\pi i n x}{2l}}$; $e^{i\frac{\pi n}{x}}$

$$f(x) \sim \sum_{-\infty}^{\infty} c_n e^{\frac{2\pi i n x}{2l}}$$

$$c_n 2l = \int_{-l}^{l} f(x) e^{-\frac{2\pi i n x}{2l}} dx$$

$$4c_n = \int_{-2}^{2} f(x) e^{-\frac{\pi i n x}{2}} dx = -\frac{2}{\pi i n} \Big]_{-2}^{2} f(x) e^{-\frac{\pi i n x}{2}} +$$

$$e^{-\frac{\pi i n x}{2}} dx = d \frac{e^{-\frac{\pi i n x}{2}}}{-\frac{\pi i n}{2}}$$

$$+ \frac{2}{\pi i n} \int_{-2}^{2} e^{-\frac{\pi i n x}{2}} f'(x)\, dx =$$

$$= -\left(\frac{2}{\pi i n}\right)^2 \Big]_{-2}^{2} e^{-\frac{\pi i n x}{2}} f'(x) +$$

$$+ \left(\frac{2}{\pi i n}\right)^2 \int_{-2}^{2} e^{-\frac{\pi i n x}{2}} d f'(x) = \left(\frac{2}{\pi i n}\right)^2 2 \cdot \left(e^{\frac{\pi i n}{2}} - e^{-\frac{\pi i n}{2}}\right) =$$

$$= -\frac{8}{\pi^2 n^2} 2i \sin\frac{\pi n}{2} = -\frac{16 i}{\pi^2 n^2} (-1)^n$$

$$4c_{2n+1} = \frac{-16 i}{\pi^2 (2n+1)^2}$$

$$c_{2n} = 0$$

$$c_{2n+1} = \frac{-4i}{\pi^2 (2n+1)^2} \cdot (-1)^n$$

$$c_0 = c_{\pm 2} = c_{\pm 4} = 0$$

$$c_1 = \frac{4i}{\pi^2} ; \quad c_3 = -\frac{4i}{9\pi^2} ; \quad c_5 = +\frac{4i}{25\pi^2}$$

$$c_{-1} = \frac{-4i}{\pi^2} ; \quad c_{-3} = \frac{4i}{9\pi^2} ; \quad c_{-5} = -\frac{4i}{25\pi^2}$$

$$c_n = \begin{cases} 0 & n \text{ jämnt} \end{cases}$$

$$c_{2n} = 0 ; \quad c_1 = \frac{4}{i\pi^2} \quad c_3 = -\frac{4}{i\pi 9} , \quad c_5 = \frac{4}{i\pi \cdot 25}$$

$$c_{-n} = \bar{c}_n = -c_n$$

$$\psi(t) = \frac{8}{\pi^2} \left\{ \sin\frac{\pi}{2} t - \frac{1}{9} \sin\frac{3\pi}{2} t + \frac{1}{25} \sin\frac{5\pi}{2} t - \right\}$$

$$\psi(t) = \sum_{-\infty}^{\infty} c_n e^{in\frac{\pi}{2}} = \frac{4}{i\pi^2}\sum_{-\infty}^{\infty}\frac{(-1)^n}{(2n+1)^2} e^{i\frac{\pi}{2}(2n+1)} = 0 \; ; \quad c_{-n} = \bar{c}_n = -c_n$$

$$f(x) = \frac{1}{2\pi}\int_{-1}^{1} e^{itx} h(t)\,dt$$

$$f'(x) = \frac{i}{2\pi}\int_{-1}^{1} e^{itx}\, t\, h(t)\,dt = \frac{1}{2\pi}\int_{-1}^{1} h(t)\, e^{itx}\, \frac{4}{\pi^2}\left\{ e^{i\frac{\pi}{2}t} - \frac{1}{9}e^{i\frac{3\pi}{2}t} + \frac{1}{25} - e^{-i\frac{\pi}{2}t} + \frac{1}{9}e^{-i\frac{3\pi}{2}t} - \frac{1}{25}\right.$$

$$= \frac{4}{\pi^2}\sum_{n=-\infty}^{n=\infty} f\left\{x + (2n+1)\frac{\pi}{2}\right\}\frac{(-1)^n}{(2n+1)^2}$$

$$f'(x) = \frac{4}{\pi^2}\sum_{n=-\infty}^{n=\infty} f\left\{x + (2n+1)\frac{\pi}{2}\right\}\frac{(-1)^n}{(2n+1)^2}$$

$f(x_0) = 0$; $\quad f(x_0 + n\pi) = 0 \qquad x_0 \pm n\pi$

$\sin(x - x_0)$

$$\left|\frac{f(x)}{\sin(x-x_0)}\right| \le 0$$

$\operatorname{Max}|f'(x)| < \operatorname{Max}|f(x)|$ $\quad$ utom $f(x) = C\sin(x - x_0)$

$$f_\varepsilon(x) = f\{(1-\varepsilon x)\left(\frac{\sin\frac{\varepsilon}{2}x}{\frac{\varepsilon}{2}x}\right)^2$$

$$|f\{(1-\varepsilon)(x+iy)\}| \le A\cdot e^{(1-\varepsilon)|y|}$$

$$\left|\frac{\sin\frac{\varepsilon}{2}(x+iy)}{\frac{\varepsilon}{2}(x+iy)}\right|^2 \le \frac{B}{|z|^2}e^{\varepsilon|y|}$$

$$\frac{\pi^2}{4} f'\left(x - \frac{\pi}{2}\right) = \cdots + \frac{1}{9}f(x-2\pi) - f(x-\pi) + f(x) - \frac{1}{9}f(x+\pi) + \frac{1}{25}f(x+2\pi) \cdots$$

$$f'\left(x - \frac{\pi}{2}\right) = \frac{\pi^2}{4}\left\{ \cdots + \frac{1}{9}f(x-2\pi) - f(x-\pi) + f(x) - \frac{1}{9}f(x+\pi) + \frac{1}{25}f(x+2\pi)\right\}$$

$M = \sup|f(x)|$; $\quad f(x_0) = M$

$f(x_0+\pi) = -M$

$f(x_0+2\pi) = M$

$$f(x) - M\cos(x - x_0) = 0 \qquad x = x_0 \pm n\pi$$

$$\left|\frac{f(x) - M\cos(x-x_0)}{\sin(x-x_0)}\right| \le K.$$

$\because f(x) = M\cos(x-x_0) + C\sin(x-x_0)$

$f(x_0) = C = 0$; $\because C = 0$

$\therefore \boxed{f(x) = M\cos(x - x_0)}$ om f reell.

Mock komplexa fal ; $\quad M = a + ib$

$C = \alpha + i\beta$

$$|f(x)|^2 = |(a+ib)\cos(x-x_0) + (\alpha+i\beta)\sin(x-x_0)|^2 =$$

$$= (a\cos(x-x_0) + \alpha\sin(x-x_0))^2 + (b\cos(x-x_0) + \beta\sin(x-x_0))^2$$

$$\left\{\frac{d}{dx}|f(x)|^2\right\}_{x=x_0} = 2(a\alpha + b\beta) = 0$$

$$\left|\frac{f(x)}{M}\right| = |\cos(x-x_0) + (a+ib)\sin(x-x_0)| =$$

$$= \left|1 - \frac{t^2}{2} + \cdots + (a+ib)t - (a+ib)\frac{t^3}{6}\right|$$

$$= \sqrt{\left(1 + at - \frac{t^2}{2}\right)^2 + b^2t^2}$$

Om $f'(x_0 - \frac{\pi}{2}) = A$ = maximal modul

$|f(x_0 + n\pi)| = |A|$ $\quad n = 0, \pm 1, \pm 2, \ldots$

$f(x_0) = A_1$
$f(x_0 + \pi) = -A_1$
$f(x_0 + 2\pi) = A_1$
$\Big\}$ $f(x_0 + n\pi) = (-1)^n A_1$

$A = A_1$ inntill i reciens cirkelbåge.

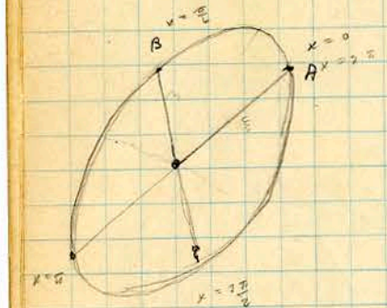
$f(x_0 + x) = A\cos x$ för $x = n\pi$, $n = 0, \pm 1, \ldots$

$\dfrac{f(x_0 + x) - A\cos x}{\sin x}$ hel funktion begränsad i hela planet

$\therefore \equiv B$

$w = f(x_0 + x) = A\cos x + B\sin x$; $f'(x) = -A\sin x + B\cos x$

$f'(x - \frac{\pi}{2}) = A\cos x + B\sin x = f(x)$



$A = e^{i\theta} a$ $\quad \therefore a > 0$
$B = ie^{i\theta} b$ $\quad -a \le b \le a$

$f(x + x_0) = e^{i\theta}\{a\cos x + ib\sin x\}$

Sats. X $\quad \angle\cap\angle(h) = M(2\sin\frac{h}{2}, 1)$

$f(x)$: sp. intervall innehållet el. $= (-a, a)$ samt

$M = \sup|f(x)|$, ni gäller

$|f(x+h) - f(x)| \in M\angle\cap\angle\{h\}$

$a = 1$;

antag $f(x) = O\left(\dfrac{e^{i(y)}}{x^{1+\varepsilon}}\right)$.

$f(x) = \frac{1}{2\pi}\int_{-1}^{1} e^{itx} h(t)\,dt$

$f(x+h) - f(x-h) = \frac{1}{2\pi}\int_{-1}^{1} e^{itx}(e^{ith} - e^{-ith})h(t)\,dt =$

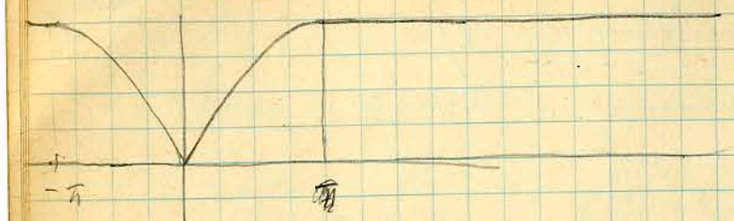
$= \frac{1}{2\pi}\int_{-1}^{1} e^{itx}\boxed{2i\sin th}\,h(t)\,dt$

$2\sin th = \frac{1}{i}\sum_{-\infty}^{\infty} a_n e^{in\frac{\pi}{2}x}$ för $-1 \le t \le 1$ konvergerar abs.

$f(x+h) - f(x-h) = \sum_{-\infty}^{\infty} a_n f(x + n\frac{\pi}{2})$

$|f(x+h) - f(x-h)| \le M \sum_{-\infty}^{\infty} |a_n(h)|$

$2\sin h \le \sum_{-\infty}^{\infty} |a_n(h)|$



$f(x) = Me^{i\lambda x}$

$f(x+h) = Me^{i\lambda x + i\lambda h} = -f(x)$ $\qquad \lambda h = \pi$

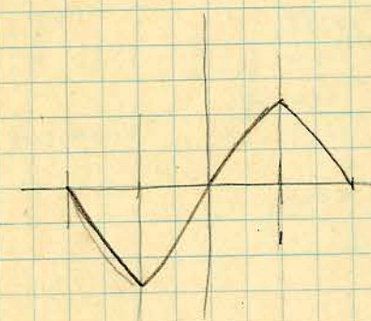
$|\lambda| \le 1$

$|h| = \frac{\pi}{\lambda} \ge \pi$

$|f(x+h) - f(x)| = 2M$ $\qquad$ für $Me^{i\frac{\pi x}{h}}$

$\psi(t) = 2\sin th \qquad -1 \le t \le 1$

$0 < h < \frac{\pi}{2}$



$$\psi(t) = \frac{1}{i}\sum_{n=-\infty}^{n=\infty} c_n(h)\, e^{(2n+1)\frac{\pi}{2}t} = \qquad c_{-n} = -c_n$$

$$= \sum_{n=1}^{\infty} 2\, c_n(h) \sin(2n+1)\frac{\pi}{2}t$$

$$2\sin h = 2\sum_{1}^{\infty} c_n(h)(-1)^n$$

Om $\underline{c_n(h)(-1)^n \ge 0}$ sehen lavirad by $\sum |a_n(h)| = 2\sum |c_n(h)|$

$= 2\sin h$

$$\varphi(t) = \psi(1+t) = 2\sum c_n(h)(-1)^n \cos(2n+1)\frac{\pi}{2}t$$

$$2(-1)^n c_n(h) = \frac{1}{2}\int_{-2}^{2} \cos(2n+1)\frac{\pi}{2}\, \varphi(t)\,dt = \int_0^2 \sin(1-t)h \cos(2n+1)\frac{\pi}{2}t\,dt$$

$2n+1 = m$

$$g_m(h) = \int_0^2 \sin(1-t)h \overbrace{\cos m\frac{\pi}{2}t\,dt}^{d\,\frac{2}{m\pi}\sin m\frac{\pi}{2}t}$$

$$g_m'(h) = -h\int_0^2 \cos(1-t)h \cos m\frac{\pi}{2}t\,dt$$

$$g_m''(h) = -h^2\int_0^2 \sin(1-t)h \cos m\frac{\pi}{2}t\,dt$$

$$g_m''(h) - h^2 g_m(h) = 0$$

$$g_m(h) = \Big[\frac{2}{m\pi}\sin(1-t)h \sin m\frac{\pi}{2}t\Big]_0^2 + \frac{2h}{m\pi}\int_0^2 \cos(1-t)h \underbrace{\sin m\frac{\pi}{2}t\,dt}_{d\,-\frac{2}{m\pi}\cos m\frac{\pi}{2}t}$$

$$= -\Big[\frac{4h}{m^2\pi^2}\cos(1-t)h \cos m\frac{\pi}{2}t\Big]_0^2 + \frac{4h^2}{m^2\pi^2}\int_0^2 \sin(1-t)h \cdot \cos m\frac{\pi}{2}t$$

$$g_m(h) = \frac{4h\cos h}{m^2\pi^2}(1-\cos m\pi) + \frac{4h^2}{m^2\pi^2}\, g_m(h)$$

$$g_m(h) = \frac{4h\cosh h\,(1-\cos m\pi)}{1-\frac{4h^2}{m^2\pi^2}}$$

$$\frac{r\sin\theta}{(r\cos\theta-\xi)^2+r^2\sin^2\theta} = \frac{r\sin\theta}{r^2+\xi^2-2r\xi\cos\theta}$$

Nollställenas fördelning hos funktioner av A-klassen.

$C_{r,x}$: $|z-x|<r$

$N(C_{r,x})$ antalet o-ställen inom $C_{r,x}$

$N(r,x) = \frac{b-a}{\pi}r + o(r)$?

$N(r,x) \le \frac{b-a}{\pi}r + K_x$?

om sp. intervallet (a,b)

Wiener-Paley

$|f(0)|=1$; sp. intervallet $=(-1,1)$ $\quad |f(x)|\le A$

$g(z) = \frac{f(z)}{A}e^{iz}$; $|g(z)|\le 1$ $\quad y\ge 0$

$|g(x)| = \left|\frac{f(x)}{A}\right|$

$$v_m(x,y) = \frac{1}{\pi}\int_{-m}^{m} \frac{\log|g(\xi)|\,d\xi}{(\xi-x)^2+y^2}$$

$v_m(x,y)$ harm. funktion till $u_m(x,y)$

$e^{u_m+iv_m} = g_m(z)$; $y>0$

$|g_m(x)| = g(x) \quad |x|<m$

$|g_m(x)| = 1 \ge |g(x)| \quad |x|>m$

$\left|\frac{g(x)}{g_m(x)}\right|\le 1$; $\left|\frac{g(z)}{g_m(z)}\right|\le 1$ i halvplanet.

$$U(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log|g(\xi)|}{(x-\xi)^2+y^2}\, d\xi$$

$f(x)$ jämn funktion med reella o-st. o med spektr. int. $= (-1,1)$

$f(0) = 1$

$$f(z) = \prod_1^{\infty}\left(1-\frac{z^2}{a_n^2}\right) \qquad a_n \text{ reella.}$$

$$\log|f(x)| = \sum \log\left|1-\frac{x^2}{a_n^2}\right| = \frac{1}{2}\int_0^{\infty} \log\left|1-\frac{x^2}{t^2}\right| d\,n(t)$$

$$\int \log\left|1-\frac{x^2}{t^2}\right| \frac{dx}{x^2} = -\frac{\log\left|1-\frac{x^2}{t^2}\right|}{x} - \frac{1}{t^2}\int \frac{2x}{1-\frac{x^2}{t^2}}\,\frac{dx}{x}$$

$$= -\frac{\log\left|1-\frac{x^2}{t^2}\right|}{x} + 2\int \frac{dx}{x^2-t^2} =$$

$$= -\frac{\log\left|1-\frac{x^2}{t^2}\right|}{x} + \frac{1}{t}\log\left|\frac{x-t}{x+t}\right|$$

$$\log|f(x)| = x^2\int_0^{\infty} \frac{n(t)}{x^2-t^2}\,\frac{dt}{t}$$

$$\frac{\log|f(x)|}{x^2} = \text{o.p.}\int_0^{\infty} \frac{n(t)}{x^2-t^2}\,\frac{dt}{t} \qquad \int_0^{x} \frac{dx}{x^2-t^2} = \frac{1}{t}\log\left|\frac{t-x}{t+x}\right|$$

$$J(x) = \int_0^{x} \frac{\log|f(x)|}{x^2}\,dx = \frac{1}{2}\int_0^{\infty} \frac{n(t)}{t}\log\left|\frac{t-x}{t+x}\right|\frac{dt}{t}$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y\log|f(\xi)|}{(x-\xi)^2+y^2}\,d\xi = u(x,y) \qquad \text{harm.} \quad y>0 \text{ och } y<0$$

$$\log|f(x+iy)| = \begin{cases} u(x,y) - ay & y \geq 0 \\ u(x,y) - by & y < 0 \end{cases}$$

$U(0,0)$

$\log|f(x+iy)| - u(x,y)$ = harm. function in $y>0$ mit $=0$ auf $\mathbb{R}$

$\delta(r,\theta) = \sum r^n (a_n \cos n\theta + b_n \sin n\theta)$; $\delta(r,-\theta) = -\delta(r,\theta)$

$$\begin{matrix} a_n \\ b_n \end{matrix} = \frac{1}{\pi} \int_0^{2\pi} \delta(r,\theta) \begin{matrix} \cos n\theta \\ \sin n\theta \end{matrix} d\theta$$

$$r^n a_n = \frac{1}{\pi} \int_0^{2\pi} \delta(r,\theta) \cos n\theta \, d\theta = 0$$

$$r^n b_n = \frac{1}{\pi} \int_0^{2\pi} \delta(r,\theta) \sin n\theta \, d\theta$$

$|\delta(r,\theta)| \le K \cdot |y| = K \cdot r|\sin\theta|$

$$|b_n| \le \frac{1}{r^n} \frac{1}{\pi} \int_0^{2\pi} K r|\sin\theta| \, d\theta \to 0 \quad n \ge 2$$

$\delta(r,\theta) = b_1 r \sin\theta = b_1 y$; $b_1 = 1$

$\max_{-\infty<x<\infty} \log|f(x+iy)| \sim y$; $y \to +\infty$

$\max_{-\infty<x<\infty} u(x,y) = o(y)$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y \log|f(\xi)|}{\xi^2 + y^2} = u(0,y)$$

$$\frac{2}{\pi} \lim_{y\to\infty} J(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log|f(\xi)|\, d\xi}{\xi^2} = \lim_{y=0} \frac{u(0,y)}{y} = \lim_{y=0} \left(\frac{\log|f(iy)|}{y} + \frac{\pi}{y} \right) = 1$$

$$\lim_{x=\infty} J(x) = \frac{\pi}{2}$$

$$\frac{\log|e^y|}{e^y} = \varphi(y) \quad ; \quad x \to e^x \; ; \; t \to e^y$$

$$\int_{-\infty}^{\infty} \varphi(y) \log \left| \frac{1+e^{x-y}}{1-e^{x-y}} \right| dy = 2 J(e^x) \to \pi$$

$$K(y) = \log \left| \frac{1+e^y}{1-e^y} \right|$$

$$\int_{-\infty}^{\infty} \varphi(x-y) K(y) \, dy \to \pi \quad x \to \infty$$

$$\int_{-\infty}^{\infty} K(y) \, dy = \frac{\pi}{2}$$

$$\int_{-\infty}^{\infty} K(y)\, e^{-iyt}\, dt = \frac{\pi}{t} \cdot \frac{e^{\frac{\pi}{2}t} - e^{-\frac{\pi}{2}t}}{e^{\frac{\pi}{2}t} + e^{-\frac{\pi}{2}t}}$$

$$\varphi_h(x) = \frac{1}{h}\int_x^{x+h} \varphi(t)\, dt \qquad \text{likv. kont. o. legs.}$$

$$\int_{-\infty}^{\infty} \left(\varphi(x-y) - \frac{2}{\pi}\right) K(y)\, dy \to 0 \quad x \to \infty$$

$$\int_{-\infty}^{\infty} \left(\varphi_h(x-y) - \frac{2}{\pi}\right) K(y)\, dy \to 0 \qquad x \to \infty.$$

$$\varphi_h(x) - \frac{2}{\pi} = \psi(x) \quad \text{legs. o. likf. kont. funktion.}$$

$$\lim_{x=\infty} \psi(x) = \infty \qquad \text{för varje } h > 0.$$

$$|\psi(x_n)| \geq d > 0$$

$$\psi(x_n + x)$$

$$\begin{cases} e^{y}\varphi(y) \geq e^{x}\varphi(x) & y \geq x \\ e^{y}\varphi(y) \leq e^{x}\varphi(x) & y \leq x \end{cases}$$

$$\varphi_h(x) = \frac{1}{h}\int_x^{x+h} \varphi(y)\, dy \geq \frac{1}{h}\int_x^{x+h} \varphi(x)\, e^{x-y}\, dy = \varphi(x)\cdot\frac{1}{h}\int_0^h e^{-h}\, dh =$$

$$= \varphi(x)\,\frac{1-e^{-h}}{h}$$

$$\varphi(y) \leq e^{x+h-y}\,\varphi(x+h)$$

$$\varphi_h(x) \leq \frac{\varphi(x+h)}{h}\int_x^{x+h} e^{x+h-y}\, dy = \varphi(x+h)\frac{e^h}{h}\cdot(1-e^{-h}) = \varphi(x+h)\,e^{\frac{h-1}{h}}$$

$$\varphi_h\frac{(x-h)he^{-h}}{e^h - e^{-h}} \leq \varphi(x) \leq \frac{h}{1-e^{-h}}\,\varphi_h(x)$$

$$\varphi_h(x) \to A \qquad x \to \infty$$

$$\begin{cases} \limsup_{x=\infty} \varphi(x) \leq A\cdot\dfrac{h}{1-e^{-h}} \\ \liminf_{x=\infty} \varphi(x) \geq A\cdot\dfrac{he^{-h}}{1-e^{-h}} \end{cases}$$

$$\varphi_h(x) - \frac{2}{\pi} = \psi(x)$$

$$x_n \to \infty \quad ; \quad |\psi(x_n)| \geq d > 0$$

$$\psi(x + x_n) = \varphi_h(x + x_n) - \frac{2}{\pi} \qquad \text{legs. o. likf. kons.}$$

$$\psi(x + x_{n_\nu}) \xrightarrow[\nu]{} g(x) \text{ likf. i varje ändligt intervall}$$

$$J_{n_\nu}(x) = \int_{-\infty}^{\infty} \{g(x-y) - \psi(x + x_{n_\nu} - y)\}\, K(y)\, dy \to 0 \qquad \nu \to \infty$$

$|g(x) - T(x+x_{n_\nu})| < \varepsilon \qquad |x| \le 2l \qquad \nu > N(\varepsilon, l)$

$|g(x-y) - T(x+x_{n_\nu}-y)| < \varepsilon \qquad |x| \le l;\ |y| \le l$

$$|J_{n_\nu}(x)| \le \varepsilon \int_{-l}^{l} |k|\,dy + \int_{|y|>l} |k|\,dy < \varepsilon'$$

$$\therefore \int_{-\infty}^{\infty} g(x-y)\,k(y)\,dy = 0$$

$$\int_{-\infty}^{\infty} g(x+\tau-y)\,k(y)\,dy = 0 \qquad -\infty < \tau < \infty$$

$P_n(x) = \sum_1^n c_\nu\, g(x+\tau_\nu)$ einen Ausdruck

Satz: mit $\varepsilon > 0$ existiert ein n und λ mit $|e^{i\lambda x} - P_n(x)| \le \varepsilon$ $\qquad -\infty < x < \infty$

$\tau < \varepsilon_n \qquad |k| \le \frac{1}{\varepsilon_n}$

$$\int_{-\infty}^{\infty} e^{i\lambda(x-y)}\,k(y)\,dy = 0$$

$$\int_{-\infty}^{\infty} e^{-i\lambda y}\,k(y)\,dy = 0$$

$$\int_{-\infty}^{\infty} e^{-ity} \ln\left|\frac{1+e^{y}}{1-e^{-y}}\right| dy = \int_0^{\infty} x^{-1-it} \ln\left|\frac{1+x}{1-x}\right| dx$$

$e^y = x \quad ; \quad \left|\frac{1+\frac{1}{x}}{1-\frac{1}{x}}\right| = \left|\frac{1+x}{1-x}\right|$

$x = \frac{1}{\xi} \quad ; \quad dx = -\frac{d\xi}{\xi^2}$

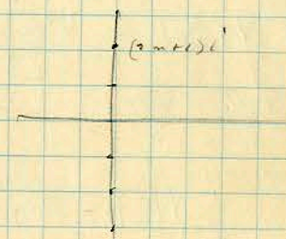
$$= \int_0^{1} x^{-1-it} \ln\left|\frac{1+x}{1-x}\right| dx + \int_1^{\infty} x^{-1-it} \ln\left|\frac{1+x}{1-x}\right| dx$$

$$\int_0^{1} \ln\left(\frac{1+x}{1-x}\right)\left\{x^{it-1} + x^{-it-1}\right\} dx =$$

$$= 2\int_0^{1} \left(x + \frac{x^3}{3} + \frac{x^5}{5}\right)\left(x^{it-1} + x^{-it-1}\right) dx =$$

$$2\left\{\left(\frac{1}{1+it} + \frac{1}{3}\cdot\frac{1}{3+it} + \frac{1}{5}\cdot\frac{1}{5+it} - \cdots\right) + \left(\frac{1}{1-it} + \frac{1}{3}\cdot\frac{1}{3-it} + \frac{1}{5}\cdot\frac{1}{5-it} + \cdots\right)\right\}$$

$$= \frac{\pi}{t}\,\frac{e^{\frac{\pi}{2}t} - e^{-\frac{\pi}{2}t}}{e^{\frac{\pi}{2}t} + e^{-\frac{\pi}{2}t}}$$



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II (44)

Primitelssatsen.

Satz IV för intervallet $x>0$; $\varphi(x)$ begr. o. $\varphi(e^x)$ likf. kont.

$|\varphi(x+y)-\varphi(x)| \le \omega\left(\frac{y}{x}\right)$; $\omega(\delta)\to 0$

Satz IV $T_\varphi = H_{\text{mgk.}}\left\{\sum c_\nu \varphi(\lambda_\nu x)\right\} \subset x^{i\lambda}$

$\vartheta(x) = \sum_{p\le x} \log p$

$\psi(x) = \vartheta(x) + \vartheta(x^{\frac12}) + \vartheta(x^{\frac13})$; $\psi(x)=0 \quad x<2$

$\sum_1^\infty \psi\left(\frac{x}{n}\right) = T(x) = \log[x]! = x\log x - x + O(\log x)$

$\frac{\psi(x)}{x} = \varphi(x)$

$\sum_1^\infty \frac{1}{n}\varphi\left(\frac{x}{n}\right) = \log x - 1 + o(1) \qquad x\to\infty$

$f(y)\Big|\ \sum_1^\infty \frac{1}{n}\varphi\left(\frac{x}{yn}\right) = \log\frac{x}{y} - 1 + o(1) \qquad \frac{x}{y}\to\infty$

$f(y)=0$; $y<1$

$\sum_1^\infty \frac{1}{n}\int_0^\infty \varphi\left(\frac{x}{yn}\right) f(y)\,dy = \int_0^\infty T_1\left(\frac{x}{y}\right) f(y)\,dy$

$\frac{1}{n}\int_0^\infty \varphi\left(\frac{x}{yn}\right) f(y)\,dy = \frac{1}{n}\int_0^\infty \varphi\left(\frac{x}{\eta}\right) f\left(\frac{\eta}{n}\right) d\eta$

$\sum \frac{1}{n^2} f\left(\frac{\eta}{n}\right) = g(\eta)$

II (45)

$\int_0^\infty \varphi\left(\frac{x}{y}\right) g(y)\,d\eta = \int_0^\infty T_1\left(\frac{x}{y}\right) f(y)\,d\eta$; $f(y)=0$, $y\le 1$.

$$f(y) = \begin{cases} \frac{2}{y^3} - \frac{1}{y^2} & y\ge 1 \\ 0 & y<1 \end{cases}$$

$g(y) = \sum_1^{[y]} \frac{1}{n^2}\left(\frac{2n^3}{y^3} - \frac{n^2}{y^2}\right) = \frac{1}{y^3}\left(\sum_1^{[y]} 2n - y\sum_1^{[y]} 1\right) =$

$= \frac{1}{y^3}\left(\frac{2([y]+1)[y]}{2} - y[y]\right) = \frac{[y]}{y^3}\left([y]+1-y\right) =$

$= \frac{[y]}{y^3}\left(1-\{y\}\right) = \underline{\frac{[y]\{-y\}}{y^3}}$

$\int_1^x T_1\left(\frac{x}{y}\right)\left(\frac{2}{y^3}-\frac{1}{y^2}\right)dy = \int_1^x \log\frac{1}{y}\, d\left(\frac{1}{y}-\frac{1}{y^2}\right) + o(1)$

$= +\int_1^x \left(\frac{1}{y^2}-\frac{1}{y^3}\right)dy + o(1) = 1-\frac12 + o(1)$

$\int_0^\infty \varphi\left(\frac{x}{y}\right) g(y)\,dy \to \frac12$; $x\to\infty$

$\varphi_k(x) = \frac{1}{\log k}\int_x^{kx} \varphi(y)\frac{dy}{y}$; $\quad y\varphi(y) \ge x\varphi(x) \quad y\ge x$

$y\varphi(y) \le kx\varphi(kx) \quad y\le kx$

$\varphi_k(x) \ge \frac{1}{\log k}\varphi(x)\cdot x\left(\frac1x - \frac1{kx}\right) = \varphi(x)\cdot\frac{k-1}{k\log k}$

$\varphi_k(x) \le \varphi(kx)\,\frac{k-1}{\log k}$

summa multiplikativ

Om $\varphi(x)$ ej konv. mot 1 gäller att $\varphi_k(x)$ · ren k tills. mot 1.

$$\int_0^\infty \varphi_k\left(\frac{x}{y}\right) g(y)\,dy \to \frac{1}{2} \qquad x \to \infty$$

$$\int_0^\infty \left(\varphi_k\left(\frac{x}{y}\right) - 1\right) g(y)\,dy \to 0 \qquad x \to \infty$$

$\psi(x) = \varphi_k(x) - 1$

$\psi'(x) = \varphi_k'(x) = \left(\frac{k\varphi(kx)}{kx} - \frac{\varphi(x)}{x}\right)\frac{1}{\log k}$

$|\psi'(x)| \le \frac{2A}{\log k}\,\frac{1}{x} = \frac{c}{x}$

$|\psi(x+y) - \psi(x)| \le c \cdot \log\left(1 + \frac{y}{x}\right)$

$|\psi(ax+ay) - \psi(ax)| \le c \cdot \log\left(1 + \frac{y}{x}\right)$

$|\psi(a_n) - 1| \ge d > 0$

$|\varphi_k(a_n) - 1| \ge d > 0$

$\varphi_k(a_n x) - 1$ delföljd av mtrlgs o'tt

$\varphi_k(a_n x) - 1$ konv. mot en punktion $\Phi(x)$

i vagi intervall $(\varepsilon, \frac{1}{\varepsilon})$

$$\int_0^\infty \Phi\left(\frac{x}{y}\right) g(y)\,dy = 0$$

$\Phi(\varphi_n x)$ lösn $\quad \sum_1^m c_n \Phi(\varphi_n x)$ lösn.

$\therefore x^{i\lambda}$ lösning

$$\int_0^\infty \left(\frac{x}{y}\right)^{i\lambda} g(y)\,dy = 0$$

$$\therefore \int_0^\infty y^{-i\lambda} g(y)\,dy = 0$$

$$\int_0^\infty y^{1-s} f(y)\,dy = \int_1^\infty \left(2y^{-s-2} - y^{-s-1}\right) dy = \frac{2}{s+1} - \frac{1}{s} =$$

$$\frac{1}{n^2}\int_0^\infty y^{1-s} f\left(\frac{y}{n}\right) dy = \frac{1}{n^s}\int_0^\infty y^{1-s} f(y)\,dy = \frac{1}{n^s}\,\frac{s-1}{s(s+1)} \qquad = \frac{s-1}{s(s+1)}$$

$$\int_0^\infty y^{s-1} g(y)\,dy = \frac{s-1}{s(s+1)} \sum_1^\infty \frac{1}{n^s} = \frac{s-1}{s(s+1)} f(s) \qquad \Re(s) > 1,$$

$$\therefore \int_0^\infty y^{-i\lambda} g(y)\,dy = \left\{\frac{s-1}{s(s+1)} f(s)\right\}_{s = 1 + i\lambda} = 0$$

$1 - s = -i\lambda$

$\therefore f(1 + i\lambda) = 0$

Wieners Tauberteorem.

$K(x)$ fix. funktion $\subset L(-\infty,\infty)$
$H(x)$ godt. " $\subset L(-\infty,\infty)$
$\varphi(x)$ godt. begr. mittlere funkt.
A godt. fal.

För att

$$\int_{-\infty}^{\infty} \varphi(x-y)\, K(y)\, dy \to A \int_{-\infty}^{\infty} K(y)\, dy \qquad x \to \infty$$

skall ledtill att

$$\int_{-\infty}^{\infty} \varphi(x-y)\, H(y)\, dy \to A \int_{-\infty}^{\infty} H(y)\, dy \qquad x \to \infty$$

för varje $H \subset L(-\infty,\infty)$ är det nödv. o. tillr. att

$$\int_{-\infty}^{\infty} e^{-itx} K(x)\, dx \neq 0 \qquad -\infty < t < \infty$$

φ : $\varphi - A$

$$\alpha(x) = \int_{-\infty}^{\infty} \varphi(x-y)\, K(y)\, dy \to 0 \qquad x \to \infty \text{ så skall}$$

$$\beta(x) = \int_{-\infty}^{\infty} \varphi(x-y)\, H(y)\, dy \to 0 \qquad x \to \infty$$

$$\alpha(x+h) - \alpha(x) = \int_{-\infty}^{\infty} \varphi(y)\, \big(K(x-y) - K(x+h-y)\big)\, dy$$

$$|\alpha(x+h) - \alpha(x)| \leq M \cdot \int_{-\infty}^{\infty} |K(y+h) - K(y)|\, dy$$

α och β likf. kont. funktioner

antag $\overline{\lim_{x=\infty}} |\beta(x)| > 0$.

$$\int_{-\infty}^{\infty} \beta(x-y)\, K(y)\, dy \to 0 \; ; \qquad x \to \infty$$

$$\beta(x-y) = \int_{-\infty}^{\infty} \varphi(x-y-\xi)\, H(\xi)\, d\xi$$

$$\underline{\int_{-\infty}^{\infty} \beta(x-y)\, K(y)\, dy} = \iint_{-\infty}^{\infty} \varphi(x-y-\xi)\, H(\xi)\, K(y)\, dy = \underline{\int_{-\infty}^{\infty} \alpha(x-\xi)\, H(\xi)\, d\xi}$$

$$|\beta(x_n)| \geq d > 0$$

$$\beta(x_n + x) \underset{\text{svgt}}{\to} \psi(x) \not\equiv 0$$

$$\int_{-\infty}^{\infty} \psi(x-y)\, K(y)\, dy = 0 \qquad \therefore \text{ exist } \lambda$$

$$\int_{-\infty}^{\infty} e^{-i\lambda y}\, K(y)\, dy = 0$$

Lemma. $f(z)$ hol. function ; $|f(0)| = 1$; $f'(0) = 0$

$$J_R = \frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z)} \frac{dz}{z}$$

$$\frac{f'(z)}{f(z)} = \sum_{a_n \in C_R} \frac{1}{z - a_n} + \text{reg. funkt.}$$

$$J_R = \sum_{a_n \in C_R} \frac{1}{a_n}$$

$$J_R = \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{z} d \ln f(z) = \underbrace{\frac{1}{2\pi i} \int_{\Gamma} \frac{1}{z} \ln f(z)}_{\frac{n(R)}{R}} + \frac{1}{2\pi i} \int_{\Gamma} \ln f(z) \frac{dz}{z^2}$$

$$\frac{dz}{iz} = \frac{Re^{i\theta} i\, d\theta}{iRe^{i\theta}} = d\theta$$

$$R \frac{1}{2\pi} \int_{\Gamma} \frac{\ln f(z)}{z} d\theta$$

$$\frac{(\ln|f| + i \arg|f|)(\cos\theta + i \sin\theta)}{R} d\theta$$

$i\, d\eta$

$$J(R) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z)} \frac{dz}{z} = \sum_{a_n \in C_R} \frac{1}{a_n}$$

$$R\{J(r)\} = \sum_i R\left(\frac{1}{a_n}\right) =$$

$$J\left(\frac{1}{z} + \frac{z}{R^2}\right) = -\frac{1}{r} \sin\theta + \frac{r}{R^2} \sin\theta = v$$

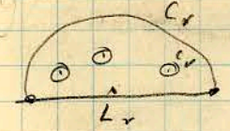
$$J_{(r)} = \frac{1}{2\pi i} \int_{\Gamma} \left(\frac{1}{z} + \frac{z}{R^2}\right) \frac{f'}{f}(z)\, dz = \frac{1}{2\pi i} \int \left(\frac{1}{z} + \frac{z}{R^2}\right) \ln f + \frac{1}{2\pi i} \int_{\Gamma} \left(\frac{1}{z^2} - \frac{1}{R^2}\right) \ln f\, dz$$

$$\int_C \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\right) ds = 0$$

$u = \ln|B(z)|$

$u = \ln|z - a| = \ln r$

$\frac{\partial u}{\partial n} = -\frac{1}{r}$



$$\frac{\partial v}{\partial n} = -\frac{1}{r^2} + \frac{1}{R^2} - \quad \text{in } L_r$$

$$\frac{\partial v}{\partial n} = -\frac{\partial v}{\partial r} = \sin\theta\left(-\frac{1}{R^2} - \frac{1}{r^2}\right) \quad \text{in } C_r$$

$$-J \sum \left(\frac{1}{a_n} + \frac{a_n}{R^2}\right) = \frac{1}{\pi R} \int_0^{\pi} \ln|f(Re^{i\theta})| \sin\theta\, d\theta + \frac{1}{2\pi} \int_{-R}^{R} \ln|f(x)| \left(\frac{1}{r^2} - \frac{1}{R^2}\right) dr$$

$f(z)$ sp. int. (a,b)

$$U(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y \lg |f(\xi)|}{(x-\xi)^2+y^2}\, d\xi$$

$$\lg|f(z)| = \begin{cases} -ay + \sum_{y(a_n)>0} \lg\left|\frac{z-a_n}{z-\bar a_n}\right| + U(z) & y>0 \\ -by + \sum_{y(a_n)<0} \lg\left|\frac{z-a_n}{z-\bar a_n}\right| + U(z) & y<0 \end{cases}$$

$$\lg|f(z)||f(-z)| = -(a-b)y + \sum_{y(a_n)>0} \lg\left|\frac{z-a_n}{z-\bar a_n}\right| + \sum_{y(a_n)<0} \lg\left|\frac{z+a_n}{z+\bar a_n}\right| + U(z)+U(-z)$$

$$f(z)f(-z) = g(z) = \prod_1^\infty \left(1-\frac{z^2}{a_n^2}\right) \quad ; \quad f(0)=1$$

$$|g(x+iy)| \le A e^{(b-a)|y|}$$

$$|g(iy)| = \prod_1^\infty \left|1+\frac{y^2}{a_n^2}\right|$$

$$|g^*(z)| = \prod_1^\infty \left(1-\frac{z^2}{|a_n|^2}\right) = \lim_{n\to\infty} g_n(z)$$

$$g_n(z) = \prod_1^n \left(1-\frac{z^2}{|a_n|^2}\right) \prod_{n+1}^\infty \left(1-\frac{z^2}{a_n^2}\right)$$

på reella axeln $|g_n(x)| \le |g(x)|$

$$\left|\frac{g_n(x+iy)}{g(x+iy)}\right| \le \prod_1^n \frac{||a_n|^2-z^2|}{|a_n^2-z^2|} \le A_n \quad |y|>\rho_n$$

$$\left.\begin{aligned} |g_n(x+iy)| &\le A_n A e^{(b-a)|y|} \\ |g_n(x)| &\le A \end{aligned}\right\} \quad |y| \ge \rho_n$$

$$|g_n(x+iy)| \le A e^{(b-a)|y|} \quad -\infty<y<\infty$$

$$|g^*(x+iy)| \le A e^{(b-a)|y|}$$

$$|g^*(iy)| \ge |g(iy)| > e^{(b-a-\varepsilon)y} \quad y \to \infty$$

$$|g(z)e^{icz}| = |g(z)|e^{-cy} \le \begin{cases} A & \text{reellt } 0<c<b-a \\ B & \text{ingen axeln} \end{cases}$$

$\le \operatorname{Max}(A,B)$ i kvadrant $\begin{cases} y>0 \\ x>0 \end{cases}$

<u>Sats. II</u> För en funktion $f \subset A$ med sp. int. (a,b) gäller för 0-st. $a_n = \alpha_n + i\beta_n \neq 0$:

1) $\sum_1^\infty \frac{|\beta_n|}{\alpha_n^2+\beta_n^2}$ konvergent

2) $n(r) \sim \frac{b-a}{\pi} r \quad r\to\infty \quad n(r) = \sum_{|a_n|<r} 1$

{3)} $\sum_1^\infty \frac{1}{a_n}$ konvergent och $=0$; $|a_n| \le |a_{n+1}|$

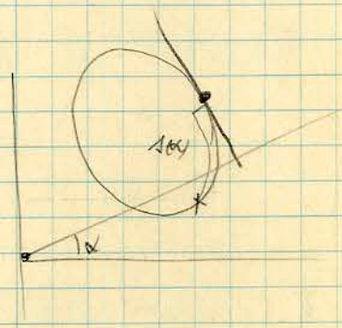
II (54)

$|f(z)| \le A e^{a|z|}$

? $\lim_{R=\infty} \frac{1}{\log R} \int_1^R \log|f(re^{i\theta})| \frac{dr}{r^2} = h(\theta)$ exist. für range θ.

1. $h(\theta)$ stützfunktion till en konvex mängd.

2. $n(r,\rho,\alpha) \sim r \frac{s(\rho)-s(\alpha)}{2\pi}$



$f(z) = \sum c_n z^n$

$g(z) = \sum \frac{c_n}{\lfloor n} z^n$

$$\log|f(re^{i\varphi})| = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log|f(\rho)|\,d\rho \quad r\sin\varphi \, d\rho}{(r\sin\varphi)^2 + (\rho - r\cos\varphi)^2} +$$

$$+ \sum_{\Im a_n > 0}^{\infty} \log\left|\frac{z-a_n}{z-\bar{a}_n}\right| - a\, r\sin\varphi$$

$$\frac{r\sin\varphi}{r^2 - 2r\rho\cos\varphi + \rho^2}$$

$$\sum_{1<r_\nu<R}\left(\frac{1}{r_\nu} - \frac{r_\nu}{R^2}\right)\cos\theta_\nu + \frac{1}{\pi R}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} V(r,\varphi)\cos\varphi\, d\varphi + \frac{1}{2\pi}\int_1^R V(r,\tfrac{\pi}{2}) + U(r,\tfrac{\pi}{2})$$

$$\left(\frac{1}{r^2} - \frac{1}{R^2}\right) dr + O(1)$$

II (56)

$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots \le \lambda_n \le \dots$

$$\overline{\lim} \frac{\sum_{\lambda_\nu < R} \frac{1}{\lambda_\nu}}{\log R} = \alpha > 0$$

$-\alpha < \arg z < \alpha \quad (1)$

$F(z)$ reg. i (1) kan approx. $\sum_{\nu=1}^{N} c_\nu z^{\lambda_\nu}$

$u_\nu(s) = R z^{\lambda_\nu}$ på Γ

$v_\nu(s) = J(z^{\lambda})$ på Γ

$$\boxed{\Phi(\lambda) = \int_\Gamma z^\lambda g(z) |dz|}$$

$z = \rho e^{i\varphi}$; $\lambda = r e^{i\theta}$

$$|\Phi(\lambda)| \le \int_\Gamma e^{R\{(\log\rho + i\varphi)(r\cos\theta + ir\sin\theta)\}} |g(z)| |dz| =$$

$$= \int_\Gamma e^{r\log\rho\cos\theta - \varphi r\sin\theta} |g| |dz|$$

$\lambda = r e^{\pm i\frac{\pi}{2}}$

$$|\Phi(\lambda)| \le e^{r\alpha'\pi} \int_\Gamma |g| |dz|$$

$$|\Phi(\lambda)| \le K' e^{Ar}$$

$$\sum_{\lambda_\nu < R} \left(\frac{1}{\lambda_\nu} - \frac{\lambda_\nu}{R^2}\right) \le \frac{2AR}{\pi R} + \frac{1}{\pi}\int_1^R \left(\frac{1}{r^2} - \frac{1}{R^2}\right) \alpha' \pi r\, dr + C' \le$$

$$\alpha' \log R + C$$

$$\overline{\lim} \frac{\sum_{\lambda_\nu < R} \left(\frac{1}{\lambda_\nu} - \frac{\lambda_\nu}{R^2}\right)}{\log R} \le \alpha' < \alpha = \frac{1}{2}$$

$0 < \beta < 1$

$$\sum_{\lambda_\nu < R} \left(\frac{1}{\lambda_\nu} - \frac{\lambda_\nu}{R^2}\right) \ge \sum_{\lambda_\nu < \beta R} \frac{1}{\lambda_\nu}(1-\beta^2)$$

$$\frac{\sum_{\lambda_\nu < \beta R} \frac{1}{\lambda_\nu}}{\log \beta R} < \frac{1}{(1-\beta^2)} \frac{1}{\left(1 + \frac{\log\beta}{\log R}\right)} \frac{\sum_{\lambda_\nu < R} \left(\frac{1}{\lambda_\nu} - \frac{\lambda_\nu}{R^2}\right)}{\log R}$$

$$\overline{\lim} \frac{\sum_{\lambda_\nu < R} \frac{1}{\lambda_\nu}}{\log R} \le \frac{1}{1-\beta} \overline{\lim} \frac{\sum_{\lambda_\nu < R} \left(\frac{1}{\lambda_\nu} - \frac{\lambda_\nu}{R^2}\right)}{\log R}$$

$\beta \to 0$

$$\overline{\lim} \frac{\sum_{\lambda_\nu < R} \frac{1}{\lambda_\nu}}{\log R} \le \overline{\lim} \frac{\sum \left(\frac{1}{\lambda_\nu} - \frac{\lambda_\nu}{R^2}\right)}{\log R}$$

$$\overline{\lim} \frac{\sum_{\nu \le R} \frac{1}{\lambda_\nu}}{\log R} > \overline{\lim} \frac{\sum_{\nu \le R} \left(\frac{1}{\lambda_\nu} - \frac{\lambda_\nu}{R^2}\right)}{\log R}$$

$$\overline{\lim} = \overline{\lim}$$

$$\overline{\lim_{R=\infty}} \frac{\sum_{\lambda_\nu \le R} \frac{1}{\lambda_\nu}}{\log R} \le \alpha' < \alpha$$

$\therefore \Phi(\alpha) \equiv 0$

$\Phi(0) = \Phi(a) - \Phi(b) - \cdots = 0$

$$H(\alpha) = \int_\Gamma \frac{g(\xi)|d\xi|}{\xi - \alpha}$$

$$H(\alpha) = -\sum_0^\infty \frac{\Phi(\nu)}{\alpha^{\nu+1}}$$

$H_2 \equiv 0$

$$\Phi(0) = \int_\Gamma g(\xi)|d\xi| = 0$$

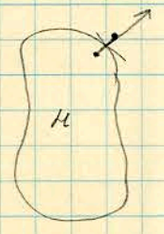
$$h(\xi) = \int_{\xi_0}^{\xi} g(\xi)|d\xi| \quad \text{reell funktion}$$

$$H(\alpha) = \int_\Gamma \frac{h(\xi)}{(\xi-\alpha)^2} d\xi$$

$H_1 = 0$

$$G_2(\alpha) = \int_\Gamma \frac{h(\xi)|d\xi|}{\xi - \alpha} = \int K_2(\alpha)\, d\alpha + C_2 \quad \text{in om } \Gamma$$

$$G_1(\alpha) = \int_\Gamma \frac{h(\xi)}{\xi - \alpha} d\xi = 0 \quad \text{utanför } \Gamma$$



$|y| \le \alpha \pi$

$G_2 \equiv 0$

$$\lim_{\varepsilon \to +0} \frac{G_2(\xi - \varepsilon e^{i\psi}) - G_1(\xi + \varepsilon e^{i\psi})}{2\pi i} = h_2(\xi)$$

$\therefore G_1(z)$ reg. anal. inom Γ med konst. randvärden $2\pi i\, h(\xi)$

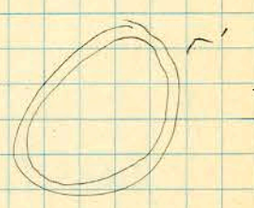
$G_1 = u_1 + i v_1$; $u_1 = 0$ på randen

$G_1 \equiv i a$; a reell konstant

$$h(\xi) = \int_{\xi_0}^{\xi} g(\xi)|d\xi| \equiv a \quad ; \quad g(\xi) = 0 \text{ m.ö. på } \Gamma.$$

$F(z)$ reg. inom och på Γ , $\varepsilon > 0$

Γ' inom Γ



$$\int_\Gamma \left| F(z) - \sum_1^n c_\nu z^{\lambda_\nu} \right|^2 |dz| < \varepsilon^2$$

$$|F(z) - P(z)| = \frac{1}{2\pi i} \left| \int_\Gamma \frac{F(t) - P(t)}{t - z} dt \right| \le \frac{1}{2\pi} \sqrt{\int_\Gamma |F(t) - P(t)|^2 |dt|} \sqrt{\frac{|dt|}{|t-z|^2}}$$

$$\le K_{z,\Gamma} \cdot \varepsilon$$

Müntz Teorem. x^{λ_ν}, $0<\lambda_\nu$, slutet på $L^2(0,1)$

endast om $\sum \frac{1}{\lambda_\nu}$ divergent.

$$\Phi(\lambda)=\int_0^1 h(x)\,x^\lambda\,dx; \qquad R(\lambda)\geq 0 \qquad |\Phi(\lambda)|\leq\sqrt{\int_0^1 |h^2(x)|\,dx}$$

$\Phi(\lambda_\nu)=0$;

$$\sum_{\lambda_\nu<R}\left(\frac{1}{\lambda_\nu}-\frac{h_\nu}{R^\nu}\right) < M$$

$\sum \frac{1}{\lambda_\nu}$ konvergent

Om x^{λ_ν} ej. slutet serien konvergent

Om serien konvergent så är x^{λ_ν} ej slutet.

$x=e^{-t}$ $\qquad h(e^{-t})e^{-\frac{t}{2}}=f(t)$; $\int_0^\infty |f^2(t)|\,dt=\int_0^1 |h^2(x)|\,dx$

$\mu_\nu=\lambda_\nu+\frac{1}{2}$

$$\int_0^\infty f(t)e^{-\lambda t}\,dt=\Phi(\lambda)$$

$$\int_0^1 h(x)\,x^\lambda\,dx=\int_1^\infty h(e^{-t})e^{-t(1+\lambda)}\,dt=\int_1^\infty f(t)e^{-(\lambda+\frac{1}{2})t}\,dt$$

$$\Phi(\lambda)=\frac{\prod_1^\infty\left(1-\frac{\lambda}{\mu_\nu}\right)}{\prod_1^\infty\left(1+\frac{\lambda}{\mu_\nu}\right)}\,\frac{1}{1+\lambda^2}$$

$$f(s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ist}\Phi(it)\,dt=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}e^{s\lambda}\Phi(\lambda)\,d\lambda$$

$s\leq 0$ $\qquad f(s)=0$

$$m(r)=\int\frac{n(r)}{r}\,dr \sim Ar$$

$$m(kr)-m(r)=\int_r^{kr}\frac{n(r)}{r}\,dr \sim A(k-1)r$$

$$n(r)\cdot\log K\leq\int_r^{k(r)}\frac{n(r)}{r}\,dr\leq n(Kr)\cdot\log K$$

$$\overline{\lim}\,\frac{n(r)}{r}\leq A\cdot\frac{K-1}{\log K}$$

$$\underline{\lim}\,\frac{n(r)}{r}\geq A\cdot\frac{K-1}{K\cdot\log K}$$

$$\int_0^r\frac{n(r)}{r}\,dr\sim A\cdot r^\sigma$$

$$\int_r^{Kr}\frac{n(r)}{r}\,dr\sim A\left((kr)^\sigma-r^\sigma\right)=r^\sigma\cdot A(K^\sigma-1)$$

$$n(r)\cdot\log K\leq\int_r^{Kr}\frac{n(r)}{r}\,dr\leq n(Kr)\cdot\log K$$

$(1+\varepsilon)^\sigma-1$

$$\overline{\lim}\,\frac{n(r)}{r^\sigma}\leq\frac{A(K^\sigma-1)}{\log K}=\frac{A\left((1+\varepsilon)^\sigma-1\right)}{\log(1+\varepsilon)} \qquad K=1+\varepsilon$$

$$\underline{\lim}\,\frac{n(r)}{r^\sigma}\geq\frac{A(K^\sigma-1)}{K^\sigma\log K}$$

$$\ln|f(z)| = \begin{cases} -ay + \sum_{y(a_n)>0} \ln\left|\frac{z-a_n}{z-\bar{a}_n}\right| + U(z) & y>0 \\ -by + \sum_{y(a_n)<0} \ln\left|\frac{z-a_n}{z-\bar{a}_n}\right| + U(z) & y<0 \end{cases}$$

$a_n = \alpha_n + i\beta_n$) $\sum \frac{|\beta_n|}{\alpha_n^2+\beta_n^2}$ konvergiert

) $U(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y \ln|f(\xi)|\,d\xi}{y^2+(x-\xi)^2}$

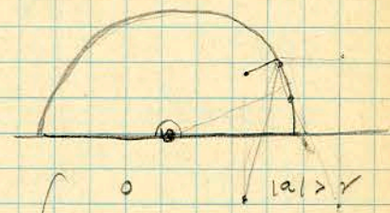
$$\frac{1}{2\pi}\int_0^{2\pi} \ln\left|\frac{re^{i\theta}-a}{re^{i\theta}-\bar{a}}\right| d\theta = 0$$

$U_2(-\theta) = U_2(\theta)$; $\frac{1}{2\pi}\int_0^{2\pi} U_1(r,\theta)\,d\theta \equiv 0$

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{r\sin\theta}{r^2 - 2\xi r\cos\theta + \xi^2}\, d\theta = 0$$

$$\frac{1}{2\pi}\int_0^{\pi} \ln\left|\frac{re^{i\theta}-a}{re^{i\theta}-\bar{a}}\right| d\theta = -u$$

$$R\left\{\frac{1}{2\pi i}\int \ln\left(\frac{z-a}{z-\bar{a}}\right)\frac{dz}{z}\right\} =$$



$$= R\left\{\frac{1}{2\pi i}\int \ln(z-a)\frac{dz}{z}\right\} = \begin{cases} \\ \\ \end{cases}$$

$$U = \frac{1}{2\pi}\int \ln|\,|\,\frac{dz}{iz} = R\left\{\frac{1}{2\pi i}\int \ln\left(\frac{z-a}{z-\bar{a}}\right)\frac{dz}{z}\right.$$

$x = r\cos\theta$
$y = r\sin\theta$

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|y|\ln|f(\xi)|\,d\xi}{y^2+(x-\xi)^2} = \frac{r}{\pi}\int_{-\infty}^{\infty} \frac{|\sin\theta|\ln|f(\xi)|\,d\xi}{r^2+\xi^2-2r\xi\cos\theta}$$

$$\frac{1}{2\pi}\int_0^{2\pi} U(r,\theta)\,d\theta = \frac{1}{\pi}\int_0^{\pi} u(r,\theta)\,d\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\xi}\ln\left|\frac{r+\xi}{r-\xi}\right| \ln|f(\xi)|\,d\xi$$

$$\int_0^{\pi} \frac{r}{\pi}\,\frac{\sin\theta}{r^2+\xi^2-2r\xi\cos\theta}\,d\theta = \int_0^{\pi} \frac{r}{\pi}\,\frac{1}{2r\xi}\,d\ln|r^2+\xi^2-2r\xi\cos\theta| =$$

$$= \frac{1}{2\pi\xi}\ln\frac{r+\xi}{|r-\xi|}$$

$$\frac{1}{2\pi}\int_0^{\pi} \ln\left|\frac{re^{i\theta}-a}{re^{i\theta}-\bar{a}}\right| d\theta = \frac{1}{2\pi}\int_0^{\pi} \ln\left|\frac{1-\rho e^{-i(\theta-\psi)}}{1-\rho e^{-i(\theta+\psi)}}\right| d\theta =$$

$|a|<r$

$a = |a| e^{i\psi}$

$\frac{a}{r} = \rho < 1$

$$\ln(1-\rho e^{-i(\theta-\psi)}) = -\sum_1^{\infty} \frac{\rho^n}{n} e^{-in(\theta-\psi)}$$

$$\ln\left(\frac{1}{1-\rho e^{-i(\theta+\psi)}}\right) = \sum \frac{1}{n} e^{-in(\theta+\psi)}$$

$$\ln(1-z) = -\sum_1^{\infty} \frac{z^n}{n}$$

$$R\,\frac{1}{2\pi}\int_0^{\pi} \sum_1^{\infty} \frac{\rho^n}{n} e^{-in\theta}\left(e^{-in\psi} - e^{in\psi}\right) d\theta$$

$$= R\left\{\frac{1}{2\pi}\sum_1^{\infty} \frac{\rho^n}{n}\left(e^{-in\psi} - e^{in\psi}\right)\frac{e^{-in\pi}-1}{-in}\right\}$$

($e^{-in\pi} = (-1)^n$)

$$R\left\{\frac{1}{2\pi}\sum \frac{\rho^n}{n}\left(e^{-in\psi} - e^{in\psi}\right)\frac{2}{in}\right\} =$$

$$-\frac{2}{\pi}\sum \frac{\rho^{2n+1}}{(2n+1)^2}\sin(2n+1)\psi$$

$$J(\psi) = \frac{1}{2\pi}\int_0^{\pi} \ln\left|\frac{1-\rho e^{-i(\theta-\psi)}}{1-\rho e^{-i(\theta+\psi)}}\right| d\theta =$$

$$R\left\{\frac{1}{2\pi}\int_0^{\pi} \sum_1^{\infty} \frac{\rho^n}{n} e^{-in\theta}\left(e^{-in\psi} - e^{in\psi}\right) d\theta\right.$$

$$J'(\psi) = R\left\{\frac{1}{2\pi}\int_0^{\pi} \sum_1^{\infty} \frac{\rho^n}{n} e^{-in\theta}\left(-e^{-in\psi} - e^{in\psi}\right) i n \, d\theta\right\} =$$

$$i \cdot -i \cdot \left(e^{in\psi} + e^{-in\psi}\right) = -2i\cos n\psi$$

$$J'(\psi) = R\left\{\frac{1}{2\pi}\sum_1^{\infty} \rho^n \cdot 2i\cos n\psi \,\frac{(-1)^n - 1}{-in}\right\} =$$

$$-\frac{2}{\pi}\sum \frac{\rho^n}{n}\cos n\psi$$

n addn

Sats. II $\quad 1/\ x^2\psi(x)$ växande funktion $x>0$

$$\text{s/}\ \int_0^\infty \varphi(x)\,dx < \infty$$

$$\mu(x) = x\int_x^\infty \psi(x)\,dx = o(x)$$

$$U(z) = \int_0^\infty \log\left|1-\frac{z^2}{t^2}\right| d\mu(t)$$

$$U(x) \geq \log 4 \cdot x^2\psi(x); \quad = \left(-x^2\frac{d}{dx}\left\{\frac{\mu(x)}{x}\right\}\right)$$

$$\mu(x) = x\int_x^\infty \psi(x); \quad \frac{d}{dx}\left\{\frac{\mu(x)}{x}\right\} = -\psi(x)$$

$$U(x) = \int_0^x \log\left(\frac{x^2}{t^2}-1\right) d\mu(t) + \int_x^\infty \log\left(1-\frac{x^2}{t^2}\right) d\mu(t)$$

$$\frac{x}{t} = \frac{1}{s} \qquad\qquad \frac{x}{t} = s$$

$$\psi_1(x) = \frac{\mu(x)}{x} = \int_x^\infty \psi(x)\,dx$$

$$U(x) = 2x\int_0^1 \frac{\psi_1(sx)-\psi_1\left(\frac{x}{s}\right)}{1-s^2}\,ds$$

ψ_1 konvex ale avtagande funktion för $0<s<1$

14/11

$$\psi_1(sx) - \psi_1\left(\frac{x}{s}\right) \geq \psi_1(sx) - \psi_1(x) \geq -x(1-s)\,\psi_1'(x) =$$

$$= x(1-s)\,\psi(x)$$

$$U(x) \geq 2x^2\,\psi(x)\int_0^1 \frac{ds}{1+s} = x^2\psi(x)\log 4.$$

$$\int_0^\infty \log\left|1-\frac{z^2}{t^2}\right| dt = \pi\,|y|$$

$$\mu'(x) = -x\,\psi(x) + \int_x^\infty \psi(x)\,dx > 0$$

$$\begin{cases} \mu'(x) = -x\,\psi(x) \\ \mu(x) \text{ växande mot } \infty \\ \frac{\mu(x)}{x} \text{ avtagande mot } 0 \end{cases}$$

$$-x^2\frac{d}{dx}\left\{\frac{\mu(x)}{x}\right\} \text{ växande}$$

$$\mu(x) = x\int_x^\infty \psi(x)\,dx$$

$\frac{\mu(x)}{x}$ ~~mot 0 avtagande~~ konvex funktion avtagand mot 0 för $x\to\infty$

$$x^2\psi(x) = -x^2\frac{d}{dx}\left(\frac{\mu(x)}{x}\right) = -\frac{x^2}{x^2}\cdot(x\mu'(x)-\mu(x)) = \mu(x) - x\mu'(x)$$

$$\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| d\mu(t) \geq 4\cdot(\mu(x) - x\mu'(x)) = 4x\cdot\mu(x)\cdot\left(\frac{1}{x}-\frac{\mu'}{\mu}\right)$$

$$= 4x\,\mu(x)\,\frac{d\log}{dx}\left(\frac{x}{\mu(x)}\right)$$

$$\int_0^\infty \log\left|1-\frac{z^2}{t^2}\right| dt = \pi|y| \qquad z = x+iy$$

$$\log\left(1-\frac{z^2}{t^2}\right)$$

$$\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| dt = 0$$

$$U(x) = 2x\int_0^1 \frac{\frac{\mu(sx)}{sx} - \frac{s\mu\left(\frac{x}{s}\right)}{x}}{1-s^2}\, ds =$$

$$\int_0^\infty \log\left(1+\frac{y^2}{t^2}\right) dt = y\int_0^\infty \log\left(1+\frac{1}{s^2}\right) ds \qquad t = sy, \quad dt = y\,ds$$

$$|t^2-z^2| = |t^2-(x+iy)^2| = \sqrt{(t^2-x^2+y^2)^2+4x^2y^2}$$

$$= \sqrt{(t^2-x^2)^2+y^4+2(t^2-x^2)y^2+4x^2y^2}$$

$$y^4-2t^2y^2+2x^2y^2$$

$$u(x) \geq \varkappa(\mu(x)-x\mu'(x))$$

$$\frac{U(x)}{x^2} \geq -\varkappa\, d\frac{\mu(x)}{x}$$

$$U(x) = \int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| \mu'(t)\,dt \; ; \qquad t = sx, \quad dt = x\cdot ds$$

$$= x\int_0^\infty \log\left|1-\frac{1}{s^2}\right| \mu'(sx)\,ds$$

$$\frac{u(x)}{x^2} = 2\int_0^1 \frac{\frac{\psi_1(sx)}{x} - \frac{\psi_1\left(\frac{x}{s}\right)}{x}}{1-s^2}\, ds$$

$$\int_x^\infty \frac{\psi_1(sy)}{y}\, dy = \int_x^\infty \frac{\mu(sy)}{sy^2}\, dy$$

$$\int_x^\infty \psi_1\left(\frac{y}{s}\right)\frac{dy}{y} = \int_x^\infty \frac{s\mu\left(\frac{y}{s}\right)}{y^2}\, dy$$

$$\begin{cases} v(x) = \frac{a}{\pi}x + x\int_x^\infty \psi(x)\,dx \\ u(x,y) = \int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| dv(t) \end{cases}$$

$$\begin{cases} u(x,y) \geq a|y| + \varkappa\cdot x^2\psi(x) \\ v(x) = \frac{a}{\pi}x - x\int_x^\infty \psi(x)\,dx \\ U(x,y) \leq a|y| - \varkappa\, x^2\psi(x) \end{cases}$$

Om $\int_{-\infty}^{\infty} \frac{h(x)}{1+x^2}\,dx$ konvergent.

finner alltid hela funktionen $\neq 0$ uppfyllande olikheten, $a > 0$

$$|f(x+iy)| \le e^{a|y| - h(x)}$$ i hela planet.

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y\,2h(\xi)}{y^2+(x-\xi)^2}\,d\xi$$

$$u(x,y) > \frac{1}{\pi}\int_{x}^{\infty} \frac{y\cdot 2h(\xi)}{y^2+(x-\xi)^2}\,d\xi >$$

$$\frac{2h(x)}{\pi}\int_{x}^{\infty} \frac{y}{y^2+(x-\xi)^2}\,d\xi =$$

$\xi - x = y\cdot t$

$$\frac{2h(x)}{\pi}\int_{0}^{\infty} \frac{y^2\,dt}{y^2(1+t^2)} = h(x).$$

$$f_2(z) = e^{-iaz + 2u(z) + i2v(z)}$$

$y \ge 0$

$$|f_1(x+iy)| \le e^{ay - 2h(x)} \qquad y \ge 0$$

$$f_2(z) = \overline{f_1(\bar z)}$$

$$|f_2(x+iy)| \le e^{-ay - 2h(x)} \qquad y \le 0$$

$$g(x) = \int_{-\infty}^{\infty} f_1(x-\xi)\,f_2(\xi)\,d\xi = \int_{-\infty}^{\infty} f_2(\xi)\,f_1(x-\xi)\,d\xi$$

def. för alla x

$$\int_{-\infty}^{\infty} e^{ay - 2h(x-\xi) + 2h(\xi)}\,d\xi$$

$e^{-2h(x)}$

$e^{-2h(x)}$

$$2\left(h(x-\xi)+h(\xi)\right) \ge 2h\left(\tfrac{x}{2}\right)$$

$|f(x) - g(x)| < \varepsilon(x)$

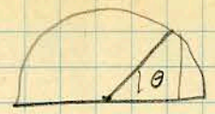
$|f(x) - f_n(x)| < \frac{\varepsilon(x)}{2}$; $\int_{-\infty}^{\infty}|f_n(x)|\,dx$; $\int_{-\infty}^{\infty}|f_n''(x)|\,dx$ konvergenta

$$\int_{-\infty}^{\infty} f_n(x)\,e^{-ixt}\,dx = h(t)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} h(t)\,e^{itx}\,dt = f_n(x)$$

$$\frac{1}{2\pi}\int_{-A(x)}^{A(x)} h(t)\,e^{itx}\,dt = g(x)$$

$\omega = (-1, 1)$



$\cos\theta = x$

$$P_n(x) = \frac{1}{2^n}\frac{1}{L^n}\frac{d^n}{dx^n}(x^2-1)^n \quad ; n = 0, 1, 2 \ldots$$

$$\frac{1}{\sqrt{1-2xz+z^2}} = \frac{1}{\sqrt{(z-e^{i\theta})(z-e^{-i\theta})}} = \sum_0^\infty P_n(x) z^n$$

$$\int_{-1}^{1} P_n P_m \, dx = \begin{cases} 0 \\ \frac{1}{n+\frac{1}{2}} \end{cases}$$

$$P_n(x) = \frac{1}{2\pi i}\int_{|z|=r<1} \frac{1}{\sqrt{1-2xz+z^2}} \frac{dz}{z^{n+1}}$$

$$P_n(x) = \frac{\sqrt{2}}{\pi}\int_0^\theta \frac{\cos(n+\frac{1}{2})t \, dt}{\sqrt{\cos t - \cos\theta}}$$

$$A(x,\xi) = \frac{\sqrt{2}}{\pi}\int_0^\theta \frac{\cos \xi t}{\sqrt{\cos t - \cos\theta}} dt$$

$$A(x, n+\tfrac{1}{2}) = P_n(x) \quad ; n = 0, 1, 2, \ldots$$

$$|A(x,\xi)| \le \frac{\sqrt{2}}{\pi} e^{|\eta|\cdot\theta}\int_0^\theta \frac{dt}{\sqrt{\cos t - \cos\theta}} < K \cdot \frac{e^{|\eta| \arccos x}}{\sqrt{1+x}}$$

$\frac{1}{\sqrt{1+\cos t}}$ $\quad \cos t = y$ $\quad \frac{1}{\sqrt{y-x}} \quad \frac{dy}{\sqrt{1-y^2}}$

$t = \arccos y$

$dt = \frac{1}{\sqrt{1-y^2}}$ $\quad \sqrt{(y-x)(1-y)(1+y)}$

$$\lim_{n=\infty} \iint_{|z|=n} \frac{A(x,z)}{\cos \pi z} \frac{dz}{z-x}$$

$\cos \pi z = 0 \quad z = \pm\frac{1}{2} \pm \frac{3}{2} \ldots \pm \frac{2n+1}{2}$

$$\frac{A(x,\xi)}{\cos \pi\xi} = \frac{2}{\pi}\sum_0^\infty (-1)^{n+1} \frac{A(x, n+\frac{1}{2})}{\xi^2 - (n+\frac{1}{2})^2}$$

$$A(x,\xi) = \frac{2\cos\pi\xi}{\pi}\sum_0^\infty \frac{(-1)^{n+1}(n+\frac{1}{2})P_n(x)}{\xi^2 - (n+\frac{1}{2})^2}$$

$$\begin{cases} \varphi_n(x) = \sqrt{n+\frac{1}{2}}\, P_n(x) & \text{norm. L. polynomet} \\ \psi_n(\xi) = (-1)^{n+1}\frac{2}{\pi}\cdot\sqrt{n+\frac{1}{2}}\,\frac{\cos\pi\xi}{\xi^2-(n+\frac{1}{2})^2} \end{cases}$$

$$A(x,\xi) = \sum_0^\infty \varphi_n(x)\psi_n(\xi)$$

Parsevals relation

$$\int_{-1}^{1} |A(x,\xi)|^2 dx = \sum_0^\infty |\psi_n(\xi)|^2$$

$$\begin{cases} \varphi_n(-x) = (-1)^n \varphi_n(x) \\ |\varphi_n(x)| \leq \sqrt{n+\frac{1}{2}} \end{cases}$$

$$\int_{-1}^{1} A(x,\xi) f(x)\,dx = \sum_0^\infty c_n \psi_n(\xi) = F(\xi)$$

$$\frac{1}{\sqrt{1-2xu+u^2}} \cdot \frac{1}{\sqrt{1-2xv+v^2}} = \sum_{n=0}^{\infty} P_n(x) P_n(x) u^n v^n$$

$$\int_{-1}^{+1} = \int_{-1}^{1}$$

$$\frac{1}{\sqrt{uv}} \log \frac{1+\sqrt{uv}}{1-\sqrt{uv}} = \sum_{n=0}^{\infty} \frac{2}{2n+1} u^n v^n$$

$$\int_{-1}^{1} P_n(x) P_m(x)\,dx = \begin{cases} 0 \\ \frac{c}{2n+1} \end{cases}$$

$P_n(1) = 1$ for $x = 1$

$(x-1)^n (x+1)^n$

$(x+1)\, D^n (x-1)^n$

$2^n L^n$

$(n+1) P_{n+1}(x) - x(2n+1) P_n(x) + n P_{n-1}(x) = 0$

$(x^2-1) y'' + 2xy' - n(n+1) y = 0$

$\left((x^2-1) y'\right)' - n(n+1) y = 0$

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$$\begin{cases} \varphi_n(x) & -1 \le x \le 1 \quad \text{De norm. Legendrelsk polynomen} \\ \psi_n(\xi) = (-1)^{n+1} \frac{2}{\pi} \sqrt{n+\frac{1}{2}} \, \frac{\cos \pi\xi}{\xi^2 - (n+\frac{1}{2})^2} & n = 0, 1, 2, \ldots \end{cases}$$

$$A(x,\xi) = \sum_0^\infty \varphi_n(x) \, \psi_n(\xi)$$

$$A(x,\xi) = \frac{\sqrt{2}}{\pi} \int_0^\theta \frac{\cos \xi t}{\sqrt{\cos t - \cos\theta}} \, dt$$

$$|A(x, \xi + i\eta)| \le K \cdot \frac{e^{|\eta| \cdot \theta}}{\sqrt{1+x}}$$

$f(x) \subset L^2(-1,1)$; $\quad c_n = \int_{-1}^1 f(x) \, \bar{\varphi}_n(x) \, dx$

$\underline{f(x) \sim \sum c_n \varphi_n(x)}$; $\quad \int_{-1}^1 \left| f(x) - \sum_1^n a_n \varphi_n(x) \right|^2 dx = \sum_1^n |c_n - a_n|^2 + \sum_{n+1}^\infty |c_n|^2$

$$E_n^2(f) = \sum_{n+1}^\infty |c_n|^2$$

$\underline{g(x) \sim \sum d_n \varphi_n(x)}$

$$\int_{-1}^1 f(x) \, \bar{g}(x) \, dx = \sum_0^\infty c_n \bar{d}_n \qquad \text{Parsevals relation}$$

$$\psi_n(m+\tfrac{1}{2}) = \begin{cases} 0 & m \ne n \\ \frac{1}{\sqrt{n+\frac{1}{2}}} & m = n \end{cases}$$

$$\int_{-1}^1 A(x,\xi) \, f(x) \, dx = \sum_0^\infty c_n \psi_n(\xi) = F(\xi) \qquad \Big| \; \xi = n + \tfrac{1}{2}$$

$$\int_{-1}^1 A(x,\xi) \, \varphi_n(x) \, dx = \psi_n(\xi)$$

$$F(n+\tfrac{1}{2}) = \frac{c_n}{\sqrt{n+\frac{1}{2}}} \le \frac{\sqrt{\sum_{n_1}^\infty |c_\nu|^2}}{\sqrt{n+\frac{1}{2}}} = \frac{E_{n-1}}{\sqrt{n+\frac{1}{2}}}$$

$$|F(n+\tfrac{1}{2})| < \frac{E_{n-1}(f)}{\sqrt{n+\frac{1}{2}}}$$

$$|F(n+3/2)| \le \frac{E_n(f)}{\sqrt{n+3/2}}$$

I Om $f \subset L^2(-1,1)$ samt $f = 0$ för $-1 < x < x_0$ n.ö. så gäller för den hela funktionen

$$F(\xi) = \sum_0^\infty c_n \psi_n(\xi) = \int_{-1}^1 A(x,\xi) \, f(x) \, dx$$

1) $|F(\xi + i\eta)| \le \text{Konst.} \; e^{|\eta| \cdot \theta_0}$; $\quad \cos\theta_0 = x_0$

2) $F(n+3/2) \le \dfrac{E_n(f)}{\sqrt{n+3/2}} \qquad n = 0, 1, 2, \ldots$

II (18)

II. Om $F(\xi)$ är en hel jämn funktion sådan att

$$|F(\xi)| = O\left(\frac{e^{|\eta|\theta_0}}{|\xi|^{1+\alpha}}\right) \quad \alpha > 1,\ 0 < \theta_0 < \pi$$

så definierar den för $-1 \le x \le 1$ abs. konvergenta serien

$$f(x) = \sum_0^\infty \sqrt{n+\tfrac{1}{2}}\, F(n+\tfrac{1}{2})\, \varphi_n(x)$$

en funktion $f(x)$ som $= 0$ för $-1 \le x \le x_0$; $\cos\theta_0 = x_0$

$$\varphi_n(x) = \sqrt{n+\tfrac{1}{2}}\, P_n(x) \qquad |P_n(x)| \le 1.$$

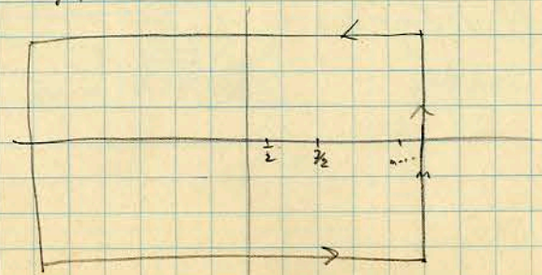
$$|\varphi_n(x)| \le \sqrt{n+\tfrac{1}{2}}$$

$\varphi_n(x)$ udda polyn. funktion om n udda
jämn " " jämnt

$$-\frac{\pi}{2}\frac{A(-x,\xi)}{\cos\pi\xi} = \sum_0^\infty \frac{\sqrt{n+\frac{1}{2}}\,\varphi_n(x)}{\xi^2-(n+\frac{1}{2})^2} \quad ; \quad \text{Res.}_{\xi = n+\frac{1}{2}} = \frac{\sqrt{n+\frac{1}{2}}\,\varphi_n(x)}{2(n+\frac{1}{2})} = \frac{1}{2}\frac{\varphi_n}{\sqrt{n+\frac{1}{2}}}$$

$$\sum_0^{n-1} \sqrt{n+\tfrac{1}{2}}\, F(n+\tfrac{1}{2})\, \varphi_n(x) =$$

$$= +\frac{1}{2\pi i}\int_{R_{l,n}} \left(-\frac{\pi}{2}\frac{A(-x,\xi)}{\cos\pi\xi}\,\xi F(\xi)\right) d\xi$$



$$\xi = \pm\eta \ ; \ \eta = \pm l$$

$$\frac{1}{2}\frac{\varphi_n(x)}{\sqrt{n+\frac{1}{2}}}\cdot(n+\tfrac{1}{2})\,F(n+\tfrac{1}{2}) \quad \text{för} \quad \xi = \pm n$$

$$|A(-x,\xi)| \le K_x e^{|\eta|(\arccos -x)}$$

$$|\cos\pi\xi| \ge \frac{1}{2} e^{\pi|\eta|}$$

$$|\xi F(\xi)| \le \frac{\text{konst.}\ e^{|\eta|\arccos x_0}}{|\xi|^\alpha}$$

$$|\ \ | \le K_x \frac{e^{-|\eta|(\pi - \arccos(-x) - \arccos x_0)}}{|\xi|^\alpha} =$$

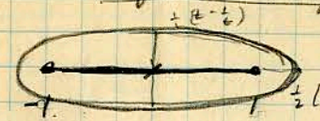
$$= K_x \frac{e^{-|\eta|(\pi - \pi + \theta - \theta_0)}}{|\xi|^\alpha} = K_x \frac{e^{-|\eta|(\theta-\theta_0)}}{|\xi|^\alpha}$$

$$\arccos -x = \pi - \theta$$

$$f(x) = \sum_0^\infty \sqrt{n+\tfrac{1}{2}}\, F(n+\tfrac{1}{2})\,\varphi_n(x) = \frac{1}{2\pi i}\int_{R_{l,\infty}} -\frac{\pi}{2}\frac{A(-x,\xi)}{\cos\pi\xi}\,\xi F(\xi)\, d\xi$$

II 80

Polynomisk approximation av funktioner.



$|P_n(x)| \leq m$ på $(-1, 1)$

$|P_n(z)| \leq m \cdot t^n$ på C_t

$z + \sqrt{z^2-1}$; på $(-1,1)$ $x \pm i\sqrt{1-x^2}$

$|z + \sqrt{z^2-1}| = 1$ på slitsen

$$\frac{|P_n(z)|}{|z+\sqrt{z^2-1}|^n} \leq m \text{ på randen}$$

$$|P_n(z)| \leq m\,|z+\sqrt{z^2-1}|^n$$

$z + \sqrt{z^2-1} = w$; $z^2-1 = (w-z)^2 = w^2 - 2wz + z^2$

$w^2 - 2wz + 1 = 0$

$$z = \frac{1}{2}(w + \frac{1}{w}) \quad ; \quad w = te^{i\theta}$$

$$z = \frac{1}{2}\left(te^{i\theta} + \frac{e^{-i\theta}}{t}\right)$$

$f(x)$ kont. på $(-1,1)$

$|f(x) - P_n(x)| \leq K \cdot \rho^n$; $\rho < 1$. $n = 0, 1, 2, \ldots$

$f(x) = P_0 + (P_1 - P_0) + (P_2 - P_1) + \ldots + (P_n - P_{n-1}) + \ldots$

kont. inuti på $(-1,1)$

$|P_n - P_{n-1}| \leq K\rho^n + K\rho^{n-1} = K'\rho^n$

$|P_n(z) - P_{n-1}(z)| \leq K'(\rho t)^n$ på ellipsen C_t

Konvergens likf. inom varje ellips C_t $t < \frac{1}{\rho}$

II (82)

Om $c_n(f) < K \cdot \rho^n$; $\rho < 1$ är funktionen analytisk på intervallet.

$f \in L^2(-1,1)$

$E_n(f) = \min \sqrt{\int_{-1}^{1} |f - P_n|^2 dx}$

$f(x) \sim \sum_0^\infty c_n \varphi_n(x)$; $P_n(x) = \sum_0^n c_n \varphi_n(x)$

$\int_0^x f(x)dx = f_1(x)$; $\int_0^x P_n(x)dx = Q_{n+1}(x)$

$|f_1(x) - Q_{n+1}(x)| \le \int_0^x |f - P_n| dx \le \sqrt{\int_0^x |f-P_n|^2 dx \int_0^x dx}$

$\le E_n(f) < K \cdot \rho^n$

$f_1(x)$ anal. reg. inom C_t ; $t\rho < 1$

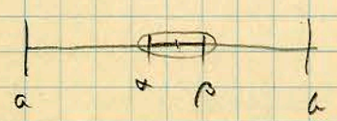
$f_1'(x)$ — — —

$f(x) = f_1'(x)$ n. ö.

Bernstein. Om $f(x)$ bomb. på (a,b) samt

$|f(x) - P_n(x)| \le K \rho^n$ för $n_1, n_2, \ldots, n_r, \ldots$

samt $f = 0$ på ett delintervall så är $f \equiv 0$ på hela (a,b).



$|P_{n_r}(x)| \le K \cdot (\rho t)^{n_r}$

$(\beta - \alpha)(\rho + \frac{1}{\rho})\frac{1}{2}$

$E_n(f, \omega) = \min \left\{ \int_\omega |f - P_n|^2 dx \right\}^{\frac{1}{2}}$ P_n av gr. n.

$E_n(f, \omega) = \sqrt{\sum_{n=1}^\infty |c_n|^2}$

Sats VII. Nödv. o. tillräckliga villkoret som en helfunktion

$E_1 \ge E_2 \ge \ldots \ge E_n \ge \ldots \to 0$ skall

uppfyllas på att relationen

1) $E_n(f, \omega) \le E_n$ $n = 0, 1, 2, \ldots$

2) $f = 0$ på ett delintervall av ω

skall leda till. $f = 0$ n. ö. på ω är att serien

$\sum_1^\infty \frac{1}{n(n-1)}$ by E_n divergent

II 84

$\rho(f,\omega) = \rho = E_1^{\frac{1}{2}} E_2^{\frac{1}{6}} E_3^{\frac{1}{12}} \cdots E_n^{\frac{1}{n(n+1)}} \cdots$

$0 \le \rho(f,\omega) \le E_1$

$\rho(f,\omega) = 0$ om $\sum_1^\infty \frac{1}{n(n+1)} \log E(f,\omega) = -\infty$

Bernsteins villkor $E_n(f,\omega) < e^{-\delta n}$ för $n = n_\nu \to \infty$

$\sum_{n_\nu}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) \log E_n < \log E_{n_\nu} \cdot \sum_{n_\nu}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{n_\nu} \log E_{n_\nu} < -\delta$

resttermen går ej mot 0 $\therefore$ serien divergerar.

$E_n(f,\omega)$

$E_n(g,\omega)$

$E_n(f+g,\omega)$

$E_n(f+g) \le \sqrt{\int_\omega |f+g-P_n-Q_n|^2 d\alpha} \le$

$\sqrt{\int_\omega |f-P_n|^2 d\alpha} \sqrt{\int_\omega |g-Q_n|^2 d\alpha} = E_n(f) \cdot E_n(g)$

Om $\rho(f,\omega)=0$ och $\rho(g,\omega)=0$ behöver ej $\rho(f+g,\omega)=0$

$\varphi(x) = \begin{cases} (x-c)^\alpha & \omega'' \\ (c-x)^\beta & \omega' \end{cases}$

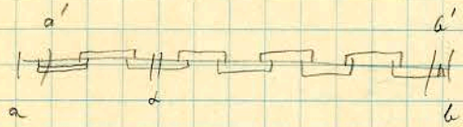
$E_n(f,\omega'') = 0$ $n \ge 1$ $\quad \rho(f,\omega'') = 0$

$E_n(f,\omega') = 0$ $n \ge 1$ $\quad \rho(f,\omega') = 0$

$E_n(f,\omega) > \frac{4}{n^2}$

$\rho(f,\omega) > 0$

1) $f \in L^2(\omega)$ till varje punkt x i ω kan tillordnas ett öppet intervall $\delta \ni x$ för vilket $\rho(f,\delta) = 0$. Om $f=0$ på ett delintervall α av ω är $f=0$ m.ö. på ω.



$E_n(g) = e^{-A_n}$, A_n mon. $\to \infty$

$E_n(f) = e^{-B_n}$; $\rho(f,\omega) = 0$ $\therefore$ $\sum_1^\infty \frac{B_n}{n(n+1)}$ divergerar

$E_n(f-g) \le \frac{1}{2}(e^{-A_n} + e^{-B_n}) \le 2e^{-\{A_n,B_n\}}$

Vilket villkor mellan talföljden B_n uppfyller för att $\sum_1^\infty \frac{B_n}{n(n+1)}$ div. leder till att $\sum \frac{\{A_n,B_n\}}{n(n+1)}$ divergerar

Nödv. o. tillr. att $\lim_{n=\infty} \frac{B_n}{n} > 0$. $\quad B_n \ge tn$, $n \ge n_0$

$\{A_n,B_n\} = \begin{cases} A_n & \text{när } A_n \le B_n \\ B_n & A_n \ge B_n \end{cases}$

1) $B_n < A_n$ $\quad n \ge N$

$\sum_N^\infty \frac{\{A_n,B_n\}}{n(n+1)} = \sum_N^\infty \frac{B_n}{n(n+1)} = \infty$.

2) $A_n > B_n$ $\quad n = n_\nu \to \infty$ $\quad \{B_n,B_{n_\nu}\}$ mon. mot ∞

$\sum_{n_\nu}^\infty \frac{\{A_n,B_n\}}{n(n+1)} \ge \{A_{n_\nu},B_{n_\nu}\} \sum_{n_\nu}^\infty \frac{1}{n(n+1)} = \frac{\{A_{n_\nu},B_{n_\nu}\}}{n_\nu} = \frac{B_{n_\nu}}{n_\nu} \ge t$

II(86)

Om $\lim \frac{A_n}{n} = 0$ kan det ena div. det andra

Till n bildens att motsv. $\varphi(n)$ = det minsta heltalet $\geq n$

för vilket $A_N > n$ $\left(A_{\varphi n} \right.$

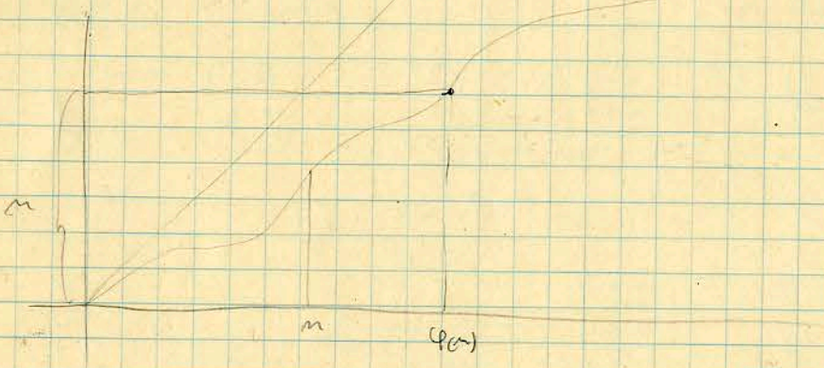
$$S_n = \begin{cases} \sum_{n}^{\varphi(n)-1} \frac{A_n}{n(n+1)} & \varphi(n) > n \\ 0 & \varphi(n) = n \end{cases}$$

Om $\lim$ man bestämma för varje $\varepsilon > 0$ godtyckligt stort tal n sådana att $S_n < 2\varepsilon$) $\frac{n}{\varphi(n)} < \varepsilon$.

$A_m < \varepsilon^2 m$ godtyckligt stort m

$n = [\varepsilon m]$ $\qquad A_{\varphi(n)} > n = [\varepsilon m]$

$\varphi(m) = \varphi([\varepsilon m]) < \varphi(m)$



$$\frac{n}{\varphi(n)} < \frac{n}{\varphi(m)} \leq \frac{[\varepsilon m]}{m} \leq \varepsilon$$

$$S_m = \sum_{n}^{\varphi(n)-1} \frac{A_n}{n(n+1)} = \sum_{n}^{m} + \sum_{m+1}^{\varphi(n)-1} < \frac{\varepsilon^2 m}{n} + \frac{m}{m+1} =$$

$$= \frac{\varepsilon^2 m}{[\varepsilon m]} + \frac{[\varepsilon m]}{m+1} < 2\varepsilon$$

Alltså kunna $n_\nu \to \infty$ väljas så att $\sum_1^\infty S_{n_\nu}$ eller $\sum_1^\infty \frac{n_\nu}{\varphi(n_\nu)}$ konvergenta

---

$n_0 = 0$ B_n definiera genom villkoren

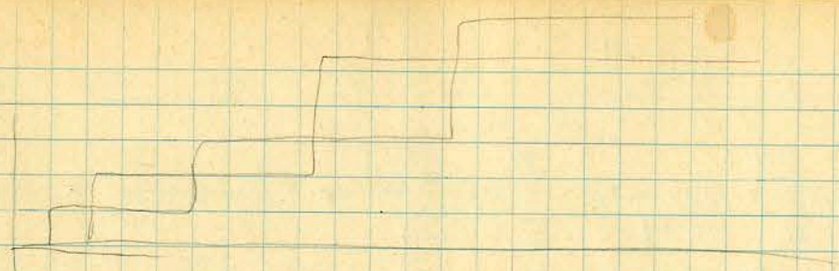
$B_n = n_\nu$; $n_\nu \leq n < n_{\nu+1}$

$\overline{\lim_{n=\infty}} \frac{B_n}{n} = 1$ eller $\sum_1^\infty \frac{B_n}{n(n+1)}$ div.

$$\sum_{n_\nu}^{n_{\nu+1}-1} \frac{\{A_n, B_n\}}{n(n+1)} = \sum_{n_\nu}^{\varphi(n_\nu)-1} + \sum_{\varphi(n_\nu)}^{n_{\nu+1}-1}$$

$\{A_m, n_\nu\} = A_m$

$A_m < n_\nu$



n^* tillordnas n vidare att n^* minsta heltal $\geq n$ för vilket $A_{n^*} > n$

Om $\lim \frac{A_n}{n} = 0$

kan man om $\varepsilon > 0$ finna godtyckligt stora tal n vidare att

$$\begin{cases} S_n = \sum_{n}^{n^*-1} \frac{A_n}{n(n+1)} < 2\varepsilon \\ \frac{n}{n^*} < \varepsilon \end{cases}$$

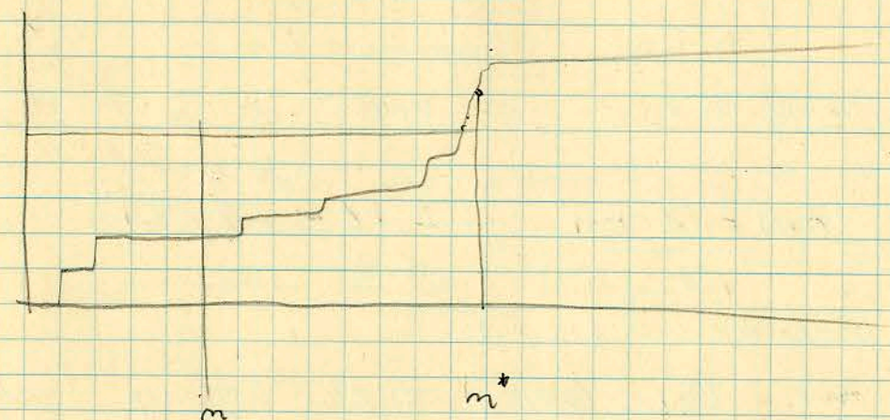
$A_n < \varepsilon^2 n$ för godtyckligt stora n

$A_n < \varepsilon n$

$A_{n^*} > n = \varepsilon n > \frac{A_n}{\varepsilon} > A_n$

$A_n n^* > n$

$\frac{n}{n^*} < \frac{n}{n} = \varepsilon$



$$S_n = \sum_{n}^{n} + \sum_{n+1}^{n^*-1} \leq \frac{\varepsilon^2 n}{n} + \frac{n}{n+1} = \varepsilon + \varepsilon \frac{n}{n+1} < 2\varepsilon$$

$\varepsilon_1 > \varepsilon_2 > \cdots \to 0$ $\sum_1^\infty \varepsilon_n$ konvergent.

$1 = n_0 < n_1 \leq n_1^* < n_2 < n_2^* < n_3 < n_3^* < n_4 < \cdots n_\nu < n_\nu^* < n_{\nu+1} <$

B_n väljas $= n_\nu$ $n_\nu \leq n < n_{\nu+1}$

$$\overline{\lim_{n=\infty}} \frac{B_n}{n} \geq \overline{\lim_{\nu=\infty}} \frac{B_{n_\nu}}{n_\nu} = 1. \quad \therefore \sum_1^\infty \frac{B_n}{n(n+1)} \text{ div.}$$

$$\sum_{n_\nu}^{n_{\nu+1}-1} \frac{\{A_n, B_n\}}{n(n+1)} = \sum_{n_\nu}^{n_\nu^*-1} + \sum_{n_\nu^*+1}^{n_{\nu+1}-1} = \underbrace{\sum_{n_\nu}^{n_\nu^*-1} \frac{A_n}{n(n+1)}}_{= S_{n_\nu} < 2\varepsilon_\nu} + \underbrace{\sum_{n_\nu^*}^{n_{\nu+1}-1} \frac{B_n}{n(n+1)}}_{< \frac{n_\nu}{n_\nu^*} < \varepsilon} < 3\varepsilon_\nu$$

$n_\nu \leq n \leq n_\nu^* - 1$
$B_n = n_\nu$
$A_n \leq n_\nu$
$\Big\} \therefore \{A_n, B_n\} = A_n$

$n_\nu^* \leq n \leq n_{\nu+1} - 1$
$B_n = n_\nu$
$A_n > n_\nu$
$\{A_n, B_n\} = B_n$

ω

$\varphi(x) = \varphi(t, \omega)$ Leg. norm. polynom

$$F(x) \sim \sum_0^\infty a_n \varphi_n(x)$$

$$b(x) \sim \sum_0^\infty b_n \varphi_n(x) \qquad S(b, \omega) = 0$$

$$E_n^2(F, \omega) = \sum_{n+1}^\infty |a_n|^2 \; ; \quad E_n^2(b, \omega) = \sum_{n+1}^\infty |b_n|^2$$

$$E(F-b, \omega) = \sum_{n+1}^\infty |a_n - b_n|^2$$

Sch. Mink. $E_n^2(F-b, \omega) \le \{ E_n(F, \omega) + E_n(b, \omega) \}^2$ allm. t.

Om b d.v.s. b_n väljas så att

$$|a_n - b_n| = |a_n| + |b_n|$$

$$E_n^2(F-b, \omega) = \sum_{n+1}^\infty (|a_n| + |b_n|)^2 > \sum_{n+1}^\infty |a_n|^2 + \sum_{n+1}^\infty |b_n|^2 =$$

$$E_n^2(F) + E_n^2(b)$$

$A_n = \log \frac{1}{E_n(F)}$; $B_n = \log \frac{1}{E_n(b)}$; $C_n = \log \frac{1}{E_n(F-b)}$

Allmänna fallet: $e^{-C_n} \le e^{-A_n} + e^{-B_n} \le 2\, e^{-\{A_n, B_n\}}$

$$C_n \ge \{A_n, B_n\} - \log 2$$

Sp. fallet: $e^{-2C_n} > e^{-2A_n} + e^{-2B_n} > e^{-2\{A_n, B_n\}}$

$$C_n < \{A_n, B_n\}$$

För att $\sum \frac{C_n}{n(n+1)}$ skall vara divergent måste

$\sum \frac{\{A_n, B_n\}}{n(n+1)}$ vara divergent för alla B_n mon. $\to \infty$

för vilka $\sum \frac{B_n}{n(n+1)}$ divergent.

$\therefore$ måste $\lim \frac{A_n}{n} = k > 0$

$$\frac{A_n}{n} = \log \left\{ \frac{1}{\sqrt[n]{E_n(F)}} \right\} > K \cdot \varepsilon_n$$

$$\sqrt[n]{E_n(F)} < e^{-K+\varepsilon_n}$$

$$\overline{\lim} \sqrt[n]{E_n(F)} < 1$$

$\therefore$ F anal. i en viss ellips. omkring ω.

(1) $\overline{\lim} \sqrt[n]{E_n(F)} = \frac{1}{R} < 1 \,, \quad R > 1$

F holomorf inom och på vissa ellipser C_ρ, $\rho < R$

Varje funktion $f \in L^2(\omega)$ kan på oändligt många sätt uppdelas i $f = f_1 + f_2$ där

$\rho(f_1, \omega) = 0$

$\rho(f_2, \omega) = 0$

$f(x) = \sum_0^\infty a_n \varphi_n(x)$

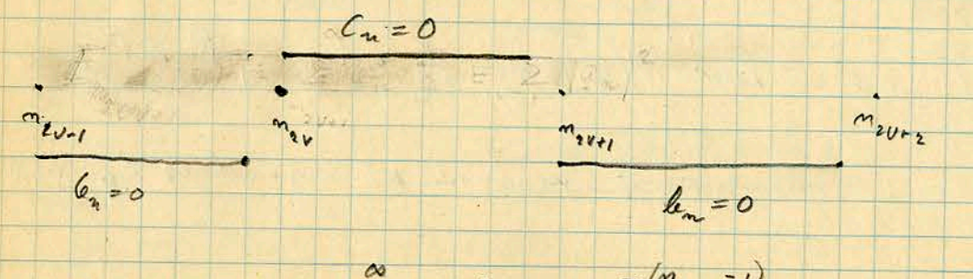
$f_1(x) = \sum_0^\infty b_n \varphi_n(x)$

$f_2(x) = \sum_0^\infty c_n \varphi_n(x)$

$0 = n_0 < n_1 < n_2 < n_3 < \dots < n_\nu < n_{\nu+1} < \dots \to \infty$

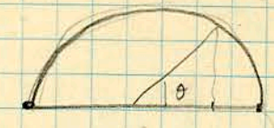
$b_n = a_n$, $c_n = 0$ $\quad n_{2\nu} \le n < n_{2\nu+1}$

$b_n = 0$, $c_n = a_n$ $\quad n_{2\nu+1} \le n < n_{2\nu+2}$



$E_{n_{2\nu+1}-1}(f_1, \omega) \le \sum_{n_{2\nu+2}}^\infty |a_n|^2 < e^{-(n_{2\nu+1}-1)}$ $\quad n_{2\nu+2}$ väljes så att

$\lim \frac{1}{n} \log \frac{1}{E_n} =$



$x = \cos\theta$

$|a_n| \le \frac{2M}{R^n}$

$g(\theta) = f(\cos\theta) \sim \sum_0^\infty a_n \cos n\theta$ $\quad a_n = \frac{2}{\pi} \int_0^\pi f(\cos\theta) \cos n\theta \, d\theta$ $\quad \cos n\theta$

$\int_0^\pi |f(\cos\theta) - \sum_0^N a_n \cos n\theta|^2 d\theta = \frac{\pi}{2} \sum_{n=1}^\infty |a_n|^2$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\cos\theta) \cos n\theta \, d\theta = \frac{1}{\pi} \int f\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right) \frac{e^{in\theta} + e^{-in\theta}}{2} d\theta$

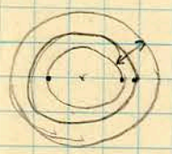
$= \frac{1}{2\pi i} \int_{|z|=1} f\left(\frac{z^2+1}{2z}\right)(z^n + z^{-n}) \frac{dz}{z}$

$z = e^{i\theta}$

$\frac{dz}{z} = i\,d\theta$

$\frac{z^2+1}{2z} = x$; $z = x \pm \sqrt{1-x^2}$

$\left| \frac{1}{2\pi i} \int \right| \le \left| \frac{1}{2\pi i} \int_{|z|=\frac{1}{R}} f\left(\frac{z^2+1}{2z}\right) z^{n-1} dz \right|$



$+ \left| \oint \frac{1}{2\pi i} \int f\left(\frac{z^2+1}{2z}\right) z^{-n-1} dz \right|$
$|z|=R$

$$\frac{1}{2\pi}\frac{M}{R^{n-1}}\,2\pi R + \frac{1}{2\pi}\frac{M}{R^{n-1}}\,\pi R = \frac{2M}{R^n}$$

$$|a_n| \le \frac{2M}{R^n}$$

$$\cos n\theta = \cos n \arccos x = T_n(x)$$

$$d\theta = \frac{dx}{\sqrt{1-x^2}}$$

$$\int_{-1}^{1} \Big| f(x) - \sum_0^n a_n T_n(x) \Big|^2 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}\sum_{n+1}^{\infty} |a_n|^2$$

$f(\cos\theta)$ $\qquad |f^{n}_{\nu}| \le A\,\frac{\gamma_n}{\gamma_1 \gamma_2 \cdots \gamma_n}$

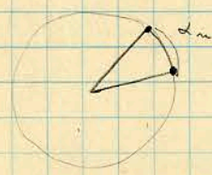
$\sum \frac{1}{\gamma_n}$ diverges $\qquad f \equiv 0$ on $f(x) = f'(x) = \cdots = 0$

$S(f_n) = 0$

$f(z) = \sum_0^{\infty} c_n z^n \qquad |f(z)| \le M. \qquad |z| < 1. \quad \text{reg. } |z| < 1.$

$\sum_1^{\infty} n |c_n| \le K < \infty$

$|f'(z)| \le K. \qquad |f(z)| \le K \cdot d(z).$ Konvex om. till $\{a_n\}$



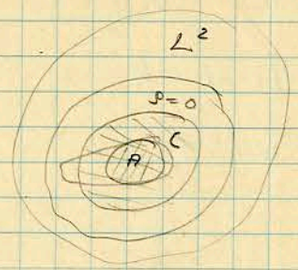
$$\int_{\alpha_n} \log |f(re^{i\theta})|\, d\theta \le \alpha_n \log K_n + \int$$

$$\le \int_0^{\alpha} \log 2\sin\frac{\theta}{2}\, d\theta < \int_0^{\alpha} \log\theta\, d\theta = \int_0^{\alpha} \theta \log\frac{\theta}{e} = \alpha \log\frac{\alpha}{e}$$

$$\log|f(re^{i\varphi})| \le \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\log|f(Re^{i\theta})|}{R^2 + r^2 - 2Rr\cos(\varphi-\theta)}\, d\theta$$

$$\int_{\alpha} \log d(\theta)\, d\theta \le 2\int_0^{\frac{\alpha}{2}} \log\theta\, d\theta = 2\int_0^{\frac{\alpha}{2}} \theta \log\frac{\theta}{e} = \alpha \cdot \log\frac{\alpha}{2e}$$

A and. p. del studio sul
6 anni i suoi lavori
B Lemma fondam.
D dem a



$$c_n = \frac{1}{2\pi i}\int_{|t|=1} f(t)\,\frac{dt}{t^{n+1}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,e^{-in\theta}\,d\theta$$

$$\varphi(z) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,e^{-iz\theta}\,d\theta \qquad |f(\theta)| < K(\pi^2-\theta^2)^{1/2} \text{ per ipotesi}$$

$$\varphi(n) = \begin{cases} c_n & n \geq 0 \\ 0 & n < 0 \end{cases} \qquad |\varphi(z)| = O\left(\frac{e^{|y|}}{|z|^{1/2}}\right) \text{ alla } \frac{1}{2}$$

$$\left|\frac{\varphi(z)}{\sin \pi z}\right| = O\left(\frac{1}{|z|^{1/2}}\right) \text{ per alla ultima}$$

$r = (n+\frac{1}{2})$

$$\frac{\varphi(n)}{\pi \cos \pi n} = \frac{\varphi(n)}{\pi}(-1)^n$$

$$\pi\frac{\varphi(z)}{\sin \pi z} = \frac{c_0}{z} - \frac{c_1}{z-1} + \frac{c_2}{z-2} + \frac{c_3}{z-3}$$

$c_n = O\left(\frac{1}{n^{1/2}}\right)$ per ipotesi $\frac{1}{2}$ $\qquad n \to \infty$

$c_0 - c_1 + c_2 - c_3 \qquad = 0$

$$\frac{a_0}{z} + \frac{a_1}{z-1} + \frac{a_2}{z-2} + \cdots + \frac{a_n}{z-n} + \cdots$$

$$\frac{1}{z-n} = \frac{1}{z} + \frac{n}{z(z-n)}$$

$$\frac{1}{z-n} = \frac{1}{z}\,\frac{1}{1-\frac{n}{z}} = \frac{1}{z} + \frac{n}{z^2} + \cdots + \frac{n^{p-1}}{z^{1/2}} + \frac{n^{1/2}}{z^{1/2}(z-n)}$$

$$\sum_0^\infty a_\nu \nu^{1/2} = 0 \quad ; \qquad \frac{1}{2} = 0, 1, 2, \ldots$$

$$g(z) = \frac{1}{2^{1/2}}\sum \frac{a_\nu \nu^{1/2}}{z-\nu} \quad ; \quad \frac{1}{2} \neq 1, 2, 3, \ldots$$

H_1

H_2

in H_2 ma H_2 gälles

$$|g(z)| \leq \frac{\sum_0^\infty |a_\nu| \nu^{1/2}}{|z^{1/2}|} = \frac{M_{1/2}}{|z|^{1/2}} \qquad \sum \frac{r^{1/2}}{M_{1/2}}$$

$$|g(z)| \leq \sum |a_\nu| \left(\frac{\nu}{|z|}\right)^{1/2} \quad \text{för varje } \frac{1}{2}$$

$$|g(z)| \le \frac{M_{12}}{|z|^{12}} \quad ; \quad M_p = \sum_0^\infty |c_\nu| \nu^{12}$$

M_0
$M_0 r_1$
$M_0 r_2$

$$|g(z)| \le M_0 \frac{r_1 r_2 \cdots r_m}{r^m} \quad ; \quad |z| = r$$

$$|g(z)| \le \frac{M_0}{m(r)}$$

$m(r) = \max \frac{r^n}{r_1 r_2 \cdots r_n}$ Största term i $1 + \frac{r}{r_1} + \frac{r^2}{r_1 r_2} + \cdots + \frac{r^n}{r_1 r_2 \cdots r_n} + \cdots$

$M_{p+1} M_{p-1} \le (M_p)^2$

$$\sum |c_\nu| \nu^{12+1} \sum |c_\nu| \nu^{12+1} \le \left(\sum |c_\nu| \nu^{12}\right)^2$$

$$\sum\sum |c_\nu| |c_\mu| \nu^{12+1} \mu^{12+1} \le \sum |c_\nu| |c_\mu| \nu^{12} \mu^{12}$$

$$\nu^{12+1} \mu^{12+1} \le \nu^{12} \cdot \mu^{12}$$

$$\frac{\mu}{\nu} \le 1$$

$$\sum\sum |c_\nu c_\mu| \nu^{12+1} \mu^{12+1}$$

$$|c_\nu c_\mu| \left(\nu^{12-1} \cdot \mu^{12+1} + \mu^{12-1} \nu^{12+1}\right) \ge 2 |c_\nu c_\mu| \nu^{12} \mu^{12}$$

$$\nu^{12} \mu^{12} \left(\frac{\mu}{\nu} + \frac{\nu}{\mu}\right) \le 2 |c_\nu c_\mu| \nu^{12} \mu^{12}$$

$M_{12} \quad \frac{M_{12}}{M_{12-1}} \le \frac{M_{12+1}}{M_{12}}$

$$\sum |c_\nu| \nu^{12} = M_{12} = M_0 \cdot r_1 r_2 \cdots r_{12}$$

$$|c_\nu| \nu^{12} \le M_0 \, r_1 r_2 \cdots r_{12}$$

$$|c_\nu| \le M_0 \cdot \frac{r_1 r_2 \cdots r_{12}}{\nu^{12}} \quad \text{för alla } 12$$

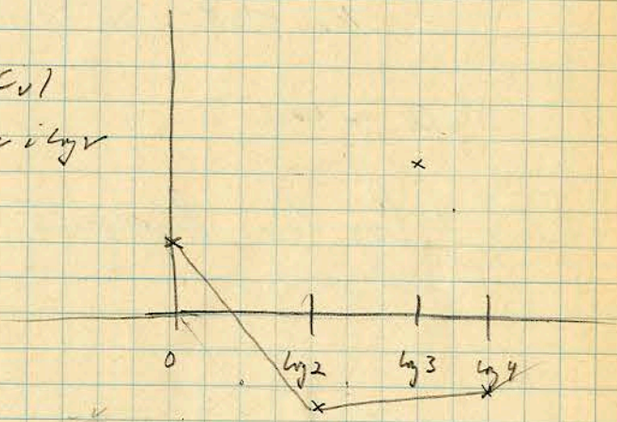
$$= \frac{M_0}{m(r)} \quad ; \quad \log m(r) \text{ konvex funktion av } \log r$$

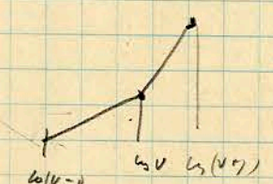
$$|c_\nu| \le \frac{M_0}{m(\nu)}$$

$$\log \left|\frac{1}{c_\nu}\right| \ge \log \frac{m(\nu)}{M_0}$$

c_ν^* den minsta maj. till $|c_\nu|$ så att $\log \frac{1}{c_\nu^*}$ är konvex i $\log \nu$

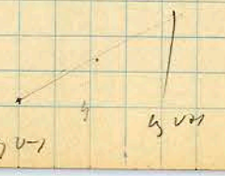
$$\frac{\log \frac{1}{c_\nu^*} - \log \frac{1}{c_{\nu-1}^*}}{\log \frac{\nu}{\nu-1}} \le$$

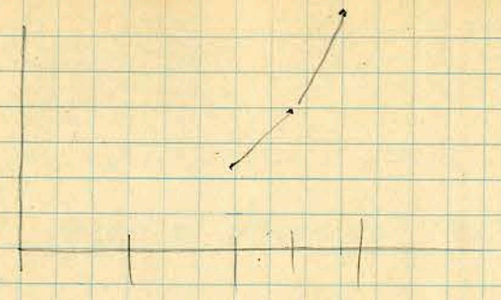




$$\frac{\log \frac{c_{\nu-1}^*}{c_\nu^*}}{\log \frac{\nu}{\nu-1}} \le \log \frac{c_\nu^*}{c_{\nu+1}^*}$$

$\log \frac{\nu+1}{\nu}$





$$\frac{A_{\nu+1} - A_\nu}{\log \frac{\nu+1}{\nu}} \geq \frac{A_\nu - A_{\nu-1}}{\log \frac{\nu}{\nu-1}}$$

$$(A_{\nu+1} - A_\nu)(\log \nu - \log(\nu-1)) \geq (A_\nu - A_{\nu-1})(\log(\nu+1) - \log \nu)$$

$$A_\nu(\log(\nu+1) - \log \nu + \log \nu - \log(\nu-1)) \geq$$

$$A_{\nu+1}(\log \nu - \log(\nu-1)) + A_{\nu-1}(\log(\nu+1) - \log \nu)$$

Nöd. villkor

$\sum_{\nu=2}^{\infty} \frac{1}{\nu} \log c_\nu^*$ divergent

END.

Just Notes. p 112–116

$$\sum_{n+1}^{2n} + \sum_{2n+1}^{4n}$$

$$\left(\frac{1}{n+1} - \frac{1}{2n+2}\right) \le \frac{1}{\nu(\nu+1)} \log A_\nu \le \left(\frac{1}{n+1} - \frac{1}{2n+2}\right) \log A_{2n}$$

$\log A_{n+1}$

$\sim \frac{1}{2n}$

$\frac{1}{\nu} - \frac{1}{\nu+1}$

$$\left(\frac{1}{n+1} - \frac{1}{n+2}\right) - \left(\frac{1}{2n+1} - \frac{1}{2n+2}\right)$$

$$\int_1^\infty \frac{u(x)}{x^2} \varphi(x)\,dx$$

$\varphi(x) = 1$ på intervallen

$$\int_1^x \varphi(x)\,dx \sim \delta x$$

$$\int_1^x \frac{u(x)}{x^2}\,dx = \psi(x) \to \infty$$

$1, -2, -4, -8, \ldots, 2^\nu, 2^{\nu+1}$

$$\frac{u(2^\nu)}{2^\nu} \le \int_{2^\nu}^{2^{\nu+1}} \frac{u(x)}{x^2}\,dx \le u(2^{\nu+1}) \int_{2^\nu}^{2^{\nu+1}} \frac{dx}{x^2} = u(2^{\nu+1}) \cdot \frac{1}{2^{\nu+1}} \cdot \frac{1}{2^{\nu+1}}$$

$$u(2^\nu) \int_{2^\nu}^{2^{\nu+1}} \frac{\varphi(x)}{x^2} \le \int_{2^\nu}^{2^{\nu+1}} \frac{u(x)}{x^2} \varphi(x)\,dx \le u(2^{\nu+1}) \int_{2^\nu}^{2^{\nu+1}} \frac{\varphi(x)}{x^2}\,dx$$

$$k(x) = \log \frac{1}{E_n} \qquad n \le x \le n + \frac{3}{2}$$

$$|F(n + 3/2)| \le E_n$$

$$|F(n + \tfrac{5}{2})| \le E_{n+1}$$

Ej $F(z)$ jämn hel funktion

1) $|F(x+iy)| \le O(e^{a|y|}$ $0 < a < \pi)$

2) $|F(n)| \le E_n$

$E_n \ge E_{n+1} \ge \ldots \to 0$

Om $\sum^\infty \frac{1}{n(n+1)} \log E_n = -\infty$ så är $f \equiv 0$.

$$F(z) = A \cdot z^p \prod \left(1 - \frac{z^2}{a_n^2}\right)$$

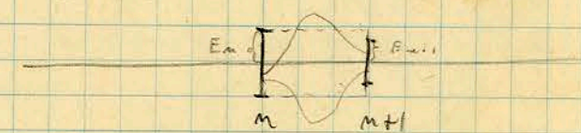
$$G(z) = |A| \cdot z^p \prod^\infty \left(1 - \frac{z^2}{|a_n|^2}\right)$$

$|G(x)| \le |F(x)|$ på reella axeln

$$|G(x+iy)| \le O(e^{a|y|})$$

$$|G'(x+iy)| = O(e^{a|y|})$$

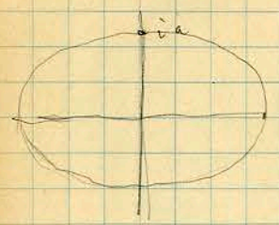
anl. o. st. på $(0, x)$ är $< \frac{a}{\pi} x$



$$k(x) = \log \frac{1}{E_n} \qquad n \le x \le n+1$$

$f(x) = \cos x + i a \sin x$; $f'(x) = -\sin x + i a \cos x$

$f'(x - \frac{\pi}{2}) = +\cos x + i a \sin x$



$|f(x)| = \max.$ pr $x = \pm n\pi$

$|f'(x)| = \max$ pr $x = \pm n\pi + \frac{\pi}{2}$

$a = a' + i a''$ $\qquad a'b' + a''b'' = 0$

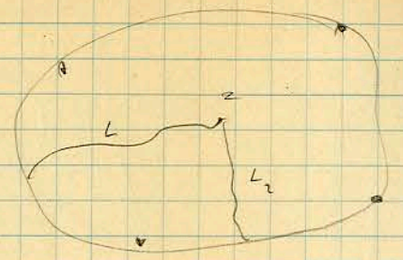
$b = b' + i b''$

$$\int_n^{n+1} \log|1/g(x)| \frac{dx}{x^2} \geq \int_n^{n+1} h(x) \frac{dx}{x^2}$$

1: a sloget om $g'(x) \neq 0$ i $(n, n+1)$ ant. $\sim (\frac{a}{\pi} - 1)x$

2: a sloget om $g'(x) = 0$ $(n, n+1)$ $\sim (\frac{\pi}{a})x$

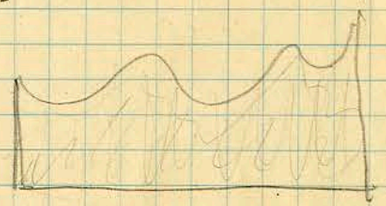
$(x, 2x)$ $\quad \int_x^{2x} \varphi(x)\, dx \sim \delta x$



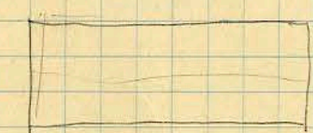
$u_\rho(z) = \inf \int_L \rho\, ds$

$v_\rho(z) = \inf \int_{L_1} \rho\, ds$

$$\begin{cases} u = u_\rho(x, y) \\ v = v_\rho(x, y) \end{cases}$$



$y = \rho^2$

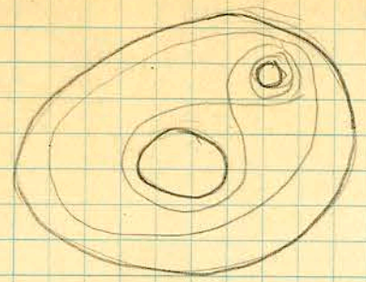


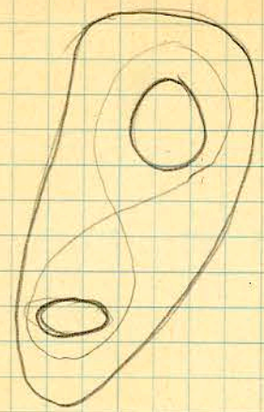
$\geq \int \rho\, d\xi \geq x$

$x^2 = \left(\int_0^a \rho\, d\xi\right)^2 \leq a \cdot \int_0^a \rho^2\, d\xi$; $y^2 \leq b \cdot \int_0^b \rho^2\, d\eta$

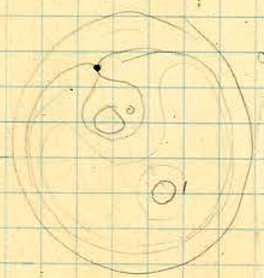
$b x^2 \leq a$ $\qquad a y^2 \leq b$

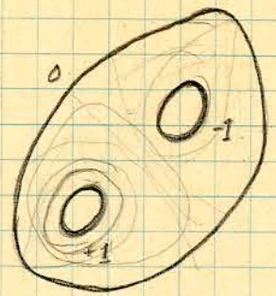
II (116)

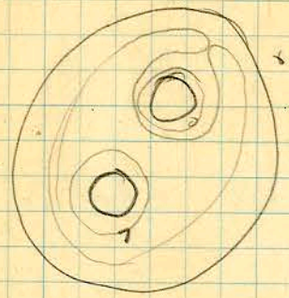




ω_1 ω_1'

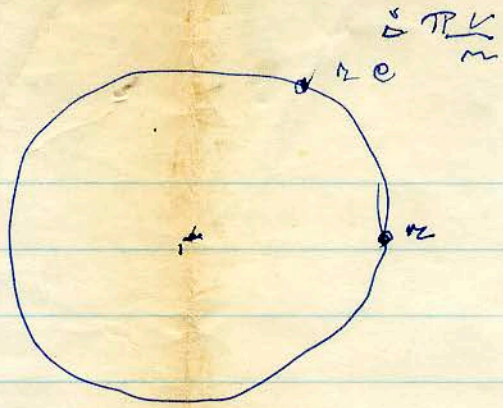






$\omega_1 - \lambda\omega_2 = 0$

$\# \; z = 2n$



$z = r \to 0$

$z = r e^{i\frac{\pi v}{n}} \to$

$$\left|\frac{z - r}{1 - z r}\right| = \left|\frac{e^{i\frac{\pi v}{n}} \cdot r}{1 - r^2 e^{i\frac{\pi v}{n}}}\right|$$

$$= \frac{r}{\left|e^{i\frac{\pi v}{n}} - r^2\right|}$$

$$\left|\frac{r e^{i\frac{\pi v}{n}} - r}{\cancel{e^{i\frac{\pi v}{n}} - r^2}\; 1 - r^2 e^{i\frac{\pi v}{n}}}\right| = \frac{r\left|1 - e^{-i\frac{\pi v}{n}}\right|}{r\left|r - \frac{1}{r}\right|}$$

$$r \left|\sin \frac{\pi v}{2n}\right|$$

$$\frac{(1-\delta_n)\left|e^{i\frac{\pi v}{n}} - 1\right|}{\left|e^{i\frac{\pi v}{n}} - (1-\delta_n)^2\right|} = \frac{\left|e^{i\frac{\pi v}{n}} - 1\right|(1-\delta_n)}{\left|e^{i\frac{\pi v}{n}} - 1 + 2\delta_n - \delta_n^2\right|}$$

$$(1-\delta_n)\frac{\left|\sin \frac{\pi v}{n}\right|}{\left|\sin \frac{\pi v}{n} + 2\delta_n - \delta_n^2\right|}$$

$$\cancel{\left|T_v(z)\right| = \left|\frac{z - r}{1 - z r}\right|}$$

$$T_v z = \frac{z - r}{1 - z r}$$