Newton Contractor of Least Square Solution Set for Overconstrained Systems

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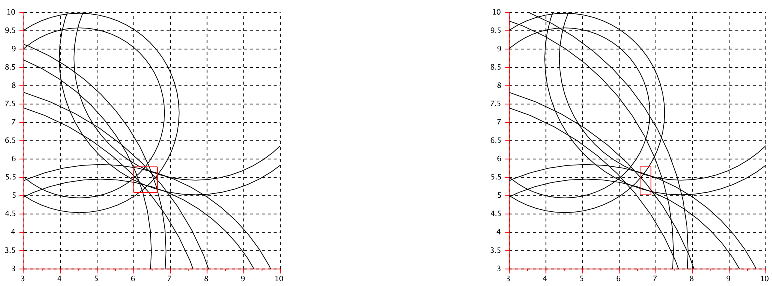
Abstract

The classical way to solve an overconstrained problem is to use the least square based methods. These methods consist to find the solution minimizing the residual errors on equations. Unfortunately, these methods require a knowledge of the uncertainties distribution and do not propose any certification on the results. The intervals, with a set approach, allow one to represent uncertainties [4], at the only and simple condition that we can bound them. We consider the overconstrained system of non-linear equation $f(x) = 0$, such that $f : \mathbb{R}^n \to \mathbb{R}^m$ with $m > n$, with uncertain values. The set of solution vanishing f is defined by $\Sigma = \{x \mid \forall k \in [1..N_C], f(x) = 0\}$. This set can be empty in case of outliers [3], and no information can be obtained. We define the set of Least Square solutions $\Sigma_{LS} = \{x \mid \forall k \in [1..N_C],\ \ argmin_x(f^2(x))\}$, and we propose a three step method to caracterize this set. The idea starts with a one-order Taylor development: $[f]_T([x]) = f(\hat{x}) + [J_f]([x]).([x] - \hat{x})$, where $\hat{x} \in [x]$ and $J_f = \frac{\partial f}{\partial x}$ the first order derivative of f. It gives us the linarized system $[A]s = [b]$ with $[A] = J_f([x])$ and $[b] = -f(\hat{x})$. It has been proven [1] that solutions are included, with certification, in $[x] \cap s + \hat{x}$ (it is the interval Newton scheme). In our case, the system $[A]s = [b]$ is overconstrained. To deal with this inconvenient, we propose to rewrite this system of equation in the second step of our method. For that, we use the Björck formula coming from the residual and a rewriting of the least square solution ($A^T Ax = A^T b$).

$$\begin{bmatrix} I & [A] \\ [A]^T & 0 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} [b] \\ 0 \end{bmatrix} \tag{1}$$

With this formula, the overestimation of the solution is lower than with the pseudo-inverse approach [2]. The problem is now reduced to a linear and well-constrained problem with interval values. This problem can be solved with the classical and powerful Hansen-Bliek-Rohn-Ning-Kearfott algorithm [1]. The contractor obtained with these three steps is included in a fix-point loop and the final algorithm provides a contractor of the set Σ_{LS}.



Example on intersection of five circles ($m = 5$ and $n = 2$). On the right picture the intersection of circles exists ($0 \in [r]$), but not on the left one (Least Square solution: $0 \notin [r]$).

References

[1] E. R Hansen. *Global Optimization Using Interval Analysis.* Marcel Dekker Inc., 2003.

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Keywords
Interval, Least Squares, Contraction