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be a bounded function on $(-\infty, \infty)$. We define the <u>harmonic transform</u> $\Phi(\sigma, t)$ of $\varphi(y)$ by

(1) $$\Phi(\sigma, t) = \int_{-\infty}^{\infty} e^{-\sigma|y| - ity} \varphi(y)\, dy$$

for $\sigma > 0$ and all real t. We wish to consider the integral equation

(2) $$\psi(x) = \int_{-\infty}^{\infty} \varphi(y)\, k(x-y)\, dy,$$

where $k \in L'$ and $\psi \in L^\infty$ are given. Let us denote the Fourier Transform of $k(x)$ by $K(t)$, and the harmonic transforms of $\varphi(x)$ and $\psi(x)$ by $\Phi(\sigma, t)$ and $\Psi(\sigma, t)$ respectively.

<u>Theorem</u>: Let $(1 + |y|^{1/2}) k(y) \in L'$. Then $\varphi \in L^\infty$ is a solution of (2) if and only if

(3) $$\frac{1}{2\pi} \int_{-\infty}^{\infty} |\Phi(\sigma, t) K(t) - \Psi(\sigma, t)|^2 dt \to 0 \quad \text{as}$$

$\sigma \to 0^+$.

<u>Proof</u>. I. Let

$$\psi_\sigma(x) = \int_{-\infty}^{\infty} e^{-\sigma|x-y|} \varphi(x-y)\, k(y)\, dy.$$

Then the F.T. of $\psi_\sigma(x)$ is $\Phi(\sigma, t) K(t)$

The F.T. of $e^{-\sigma|x|}\psi(x)$ is $\Psi(\sigma, t)$

Let $\delta_\sigma(x) = \psi_\sigma(x) - e^{-\sigma|x|}\psi(x)$.

We shall prove

(A) $\lim_{\sigma \to 0+} \delta_\sigma(x) = 0$ uniformly in x

(B) $\int_{-\infty}^{\infty} |\delta_\sigma(x)|^2 dx$ exists and $\to 0$ as $\sigma \to 0^+$.

Then, by Plancherel-Parseval, (2) $\Rightarrow$ (3) will follow.

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Then, by Plancherel-Parseval, (2) $\Rightarrow$ (3) will follow.

(A) $$\delta_\sigma(x) = \Psi_\sigma(x) - e^{-\sigma|x|}\Psi(x)$$

$$= \int_{-\infty}^{\infty}\{e^{-\sigma|x-y|} - e^{-\sigma|x|}\}\varphi(x-y)k(y)\,dy$$

If $|\varphi| < M$, then

$$|\delta_\sigma(x)| \leq M\int_{-\infty}^{\infty}|e^{-\sigma|x-y|} - e^{-\sigma|x|}|\,|k(y)|\,dy$$

$$= M\int_{-a}^{a} + M\int_{|y|>a} .$$

The first $\to 0$ uniformly as $\sigma \to 0^+$; the second $\to 0$ uniformly as $a \to \infty$; hence $\delta_\sigma(x) \to 0$ uniformly as $\sigma \to 0^+$.

(B) By Schwarz inequality and hypothesis on $k(y)$,

$$|\delta_\sigma(x)|^2 \leq CM^2\int_{-\infty}^{\infty}\{e^{-\sigma|x-y|} - e^{-\sigma|x|}\}^2|y|^{-1/2}|k(y)|\,dy$$

and

$$\int_{-\infty}^{\infty}|\delta_\sigma(x)|^2 dx \leq CM^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\quad\}^2|y|^{-1/2}|k(y)|\,dy\,dx.$$

Under $\xi = \sigma x$ and $\eta = \sigma y$,

$$\int_{-\infty}^{\infty}\{e^{-\sigma|x-y|} - e^{-\sigma|x|}\}^2 dx = \frac{1}{\sigma}\int_{-\infty}^{\infty}\{e^{-|\xi-\eta|} - e^{-|\xi|}\}^2 d\xi$$

$$\equiv \frac{1}{\sigma}f(\eta)$$

If $|\eta| < 1$, then

$$e^{-|\xi-\eta|} - e^{-|\xi|} < e^{-|\xi|}(e^{|\eta|} - 1)$$

$$< Ce^{-|\xi|}|\eta|$$

and $f(\eta) < C_0\eta^2$

If $|\eta|>1$,

$$\int_{-\infty}^{\infty}(e^{-|\xi-\eta|}-e^{-|\xi|})^2 d\xi \leq 2\int_{-\infty}^{\infty} e^{-2|\xi|}d\xi < 4.$$

Hence there exists c_1 such that for all η,

$$f(\eta) < \frac{c_1\eta^2}{1+\eta^2}.$$

Then

$$\int_{-\infty}^{\infty}|\delta_\sigma(x)|^2dx \leq CM^2\int_{-\infty}^{\infty}\frac{1}{\sigma}f(\sigma y)\,|y|^{-1/2}|k(y)|\,dy$$

$$\leq C_2\int_{-\infty}^{\infty}\frac{\sigma|y|}{1+\sigma^2y^2}\,|y|^{1/2}|k(y)|\,dy$$

Now $|y|^{1/2}|k(y)| \in L^1$, and $\frac{\sigma|y|}{1+\sigma^2y^2} \in L^\infty$ and $\to 0$ as $\sigma\to 0^+$. Hence, by dominated convergence,

$$\int_{-\infty}^{\infty}|\delta_\sigma(x)|^2dx \to 0 \quad \text{as } \sigma\to 0^+.$$

II. Suppose (3) is given, to prove (2).

Let $\Psi_1(x) = \int_{-\infty}^{\infty}\varphi(y)\,k(x-y)\,dy$.

$\varphi\in L^\infty$, $k\in L^1 \Rightarrow \Psi_1 \in L^\infty$. Hence Ψ_1 has a harmonic transform in L^2, say $\underline{\Psi}_1(\sigma,t)$. By part I,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|\underline{\Phi}(\sigma,t)K(t)-\underline{\Psi}_1(\sigma,t)|^2dt \to 0$$

as $\sigma\to 0^+$. The same holds for $\underline{\Psi}(\sigma,t)$ by hypothesis, so that by the triangle inequality in L^2,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|\underline{\Phi}(\sigma,t)-\underline{\Psi}_1(\sigma,t)|^2dt \to 0 \text{ as } \sigma\to 0^+.$$

By the Parseval relation, then,

$$\int_{-\infty}^{\infty} |\psi(x) - \psi_1(x)|^2 e^{-2\sigma|x|}\, dx \to 0$$

as $\sigma \to 0^+$. But this is a positive function of σ which increases as σ becomes smaller and yet approaches 0. Hence it is $\equiv 0$ for all σ. Thus, for example,

$$\int_{-\infty}^{\infty} |\psi(x) - \psi_1(x)|^2 e^{-|x|}\, dx = 0,$$

and $\psi = \psi_1$ a.e.

Hence (3) $\Rightarrow$ 2, and the proof is complete.

Lecture notes taken by unknown from lectures at either Bell Lab. or Harvard 48-49