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It ... applications to have a modulated U 1

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is neg def. The following results are of a similar kind:

Let $f(x)$ be ~~an even~~ a continuous positive function on R with these properties: as x increases from 0 to ∞

(1) $\frac{f(x)}{x}$ decreases steadily from some finite value to 0,

(2) $x f(x)$ increases " " 0 to ∞,

~~(3) $f(x) \in L^1(0,1)$~~.

Then there exists a neg. def. λ equivalent with $\int_0^x f(x)\,dx$ in the sense that

$$a \le \frac{\lambda(x)}{\int_0^x f(x)\,dx} \le b \quad , \quad x > 0$$

where a, b are pos. finite constants.

We shall several times use this elementary lemma:
Let $\varphi(t)$ be of finite (total) variation on (a,b) and $m = \min(|\varphi(a)|, |\varphi(b)|)$, then

$$(4) \qquad \left| \int_a^b \cos 2xt \, \varphi(t)\,dt \right| \le \frac{1}{|x|} \left(m + \int_a^b |d\varphi| \right)$$

It is useful for applications to have a modulated family of, in a sense, "regular" negative definite functions, at least in the one-dimensional case. We know already that each even and pos. $\lambda(x)$ which is concave for $x>0$ is neg def. The following results are of a similar kind:

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U(2)

Define on $(0,\infty)$

(5) $$\ell(x) = x f(x).$$

Then by assumption $\ell(x)$ is monotonic increasing.

Define on $(0,\infty)$

(6) $$\sigma(t) = t^2 \ell(\tfrac{1}{t}) = t f(\tfrac{1}{t}).$$

Then on $(0,\infty)$, $\sigma(t)$ increases from 0 to $\lim_{x\downarrow 0} \frac{f(x)}{x}$.

A neg. def. $\lambda(x)$ is now defined by

$$\lambda(x) = \int_0^\infty \frac{\sin^2 xt}{t^2}\, d\sigma(t)$$

Since $\sin^2 1 > \frac{2}{3}$ we obtain

(7) $$\lambda(x) > \frac{2}{3}\int_0^{\frac{1}{x}} x^2\, d\sigma(t) = \frac{2}{3}\, x^2 \sigma(\tfrac{1}{x}) = \frac{2}{3}\, \ell(x).$$

We also have

$$\lambda(x) > \frac{1}{2}\int_{\frac{1}{x}}^{\infty} \frac{1-\cos 2tx}{t^2}\, d[t^2 \ell(\tfrac{1}{t})] =$$

$$= \frac{1}{2}\int_{\frac{1}{x}}^{\infty} (1-\cos 2tx)\, d[\ell(\tfrac{1}{t})] + \int_{\frac{1}{x}}^{\infty} \frac{1}{t}\, \ell(\tfrac{1}{t})\, dt - \int_{\frac{1}{x}}^{\infty} \cos 2tx\, \frac{1}{t}\, \ell(\tfrac{1}{t})\, dt$$

$$= A + B + C$$

where

$$B = \int_0^x \frac{l(s)}{s}\,ds = F(x) \equiv \int_0^x f(s)\,ds$$

$$A > -\frac{1}{2}\int_{\frac{1}{x}}^{\infty} \left|d\, l\left(\tfrac{1}{t}\right)\right| = -\frac{l(x)}{2} .$$

Since $\frac{1}{t}\, l\left(\frac{1}{t}\right)$ is monotonic decreasing on $\left(\frac{1}{x}, \infty\right)$ ~~we ma~~ with variation $= x\, l(x)$ we may apply (4) to C:

$$|C| \leq \frac{1}{x}\, x\, l(x) = l(x)$$

Thus

$$(8) \qquad \lambda(x) > F(x) - \frac{3}{2}\, l(x) .$$

Setting $l/F = \theta$ we obtain on combining (7) and (8):

$$(9) \qquad \frac{\lambda(x)}{F(x)} > \max\left(1 - \frac{3}{2}\theta,\ \frac{2}{3}\theta\right) = \frac{12}{39} = \frac{4}{13}$$

On the other hand

$$\lambda(x) = \int_0^{\frac{1}{x}} + \int_{\frac{1}{x}}^{\infty} < l(x) + \int_{\frac{1}{x}}^{\infty} \frac{1}{t^2}\, d\left[t^2\, l\left(\tfrac{1}{t}\right)\right]$$

$$= 2\int_{\frac{1}{x}}^{\infty} l\left(\tfrac{1}{t}\right) \frac{dt}{t} = 2\, F(x)$$

Hence

$$\frac{4}{13} < \frac{f(x)}{F(x)} < 2$$

This inequality is of course only interesting for the behavior of $f(x)$ at $x = \infty$. If appropriate conditions are imposed on $f(x)$ on $(0,1)$ and the previous conditions on $(1,\infty)$ more general results of the same kind can be obtained.

$$l(0) = \lim \lim_{t=\infty} \frac{f(\frac{1}{t})}{t} = \lim_{x=0} f(x)\cdot x$$