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$$\ldots \varrho(x+y) - T_{\varepsilon_n} \varrho(x)\Big|^2 d\sigma(y) \quad , \quad \varepsilon_n \downarrow 0$$

Then $\varphi_1(x) \geq \varphi_2(x) \geq \cdots \geq \varphi_n(x) \geq \cdots \geq 0$

and $\varphi_n \in \mathcal{B}$, $\lim_{n\to\infty} \varphi_n(x) = 0$, a.e.

Lebesgue's theorem applied to (10) yield

$$\lim_{n\to\infty} F(\varphi_n) = 0$$

Thus $\varphi_n \xrightarrow[\mathcal{B}]{} 0$ and there exists linear convex combinations $\sum_1^m \lambda_\nu \varphi_\nu$ with arbitrary small norm.

But $\|\varphi_m\| \leq \|\sum_1^m \lambda_\nu \varphi_\nu\|$ and it follows that $\|\varphi_m\|_{\mathcal{B}} \to 0$, $\|T_{\varepsilon_n} \varrho\| \to 0$.

$$\varphi_n(x) = \int \left| T_{\varepsilon_n} \rho(x+y) - T_{\varepsilon_n} \rho(x) \right|^2 d\sigma(y) \quad , \quad \varepsilon_n \downarrow 0$$

Then $\quad \varphi_1(x) \geq \varphi_2(x) \geq \cdots \geq \varphi_n(x) \geq \cdots \geq 0$

and $\varphi_n \in \mathcal{B}$, $\lim_{n\to\infty} \varphi_n(x) = 0$, a.e.

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$\|T_{\varepsilon_n} \rho\| \to 0$.

Let $\mathcal{D}$ be a spec. Dir. space on a compact $\mathcal{G}$.

Define for open $\omega \subset \mathcal{G}$

$$A(\omega) = \sup_{\substack{u \in \mathcal{D}^+ \\ \|u\| \leq 1}} \int_\omega u^2(x)\,dx,$$

and for closed $E \subset \mathcal{G}$,

$$A(E) = \inf_{\omega \supset E} A(\omega).$$

Assume (condition I): For each $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that ~~a~~ $|\omega| < \delta \Rightarrow A(\omega) < \varepsilon$ $\left(|\omega| = \int_\omega dx \right)$.

Let $\mathcal{B} = \mathcal{B}_{\mathcal{D}}$ be the Banach space of equivalence classes of real valued Borel measurable functions $\varphi(x)$, with norm

$$\|\varphi\|_{\mathcal{B}} = \sup_{\substack{u \in \mathcal{D}^+ \\ \|u\| \leq 1}} \int u^2(x)\,|\varphi(x)|\,dx .$$

φ_1 and φ_2 belong to the same class if $\varphi_1 - \varphi_2 = 0$ a.e.

Obvious properties of $\mathcal{B}$:

(1) $$L^\infty \subset \mathcal{B} \subset L^1$$

(2) $\text{const.} \|\varphi\|_{L^1} \leq \|\varphi\|_{\mathcal{B}} \leq \|\varphi\|_{L^\infty} \cdot A(\mathcal{G})$

(3) $$\|\varphi\| = \||\varphi|\|$$

$$(4)\begin{cases} \text{If } \psi \text{ is measurable and } \varphi \in B,\ |\psi(x)| \leq |\varphi(x)| \text{ a.e.} \Rightarrow \\ \psi \in B,\ \ \|\psi\| \leq \|\varphi\| \end{cases}$$

Lemma: Under the condition

$$(5)\qquad T_n\varphi \xrightarrow[B]{} \varphi \quad \text{as } n \to \infty,\ \forall \varphi \in B,$$

the property $\|T_\varepsilon \rho\| \to 0$, $\varepsilon \downarrow 0$, holds for multipliers ρ in $\mathfrak{D}$ (T_ε and T_n = circular contractions)

It seems likely that (5) follows from condition I, but this is still unproven. In order to prove the lemma let F be a linear functional on B, and let φ_E be the char. function of a Borel set E. Then

$$|F(\varphi_E)| \leq \|F\|_{B'}\, A(E)$$

and $F(\varphi_E)$ is therefore an additive set function tending to 0 with $|E|$ and hence of the form

$$F(\varphi_E) = \int_E f(x)\,dx \quad,\ f \in L^1(\mathcal{G}).$$

Each $\varphi \in L^\infty(\mathcal{G})$ can now be approached:

$$\Big|\varphi(x) - \sum_1^n c_\nu\, \varphi_{E_\nu}(x)\Big| < \varepsilon \quad \text{a.e.}$$

$$\Rightarrow \quad \left| F\left(\varphi - \sum_{1}^{n} c_\nu \varphi_{E_\nu}\right) \right| < \varepsilon \, \|F\| \, A(\mathcal{G}) \quad \Rightarrow$$

$$(6) \qquad F(\varphi) = \int f(x)\,\varphi(x)\,dx \qquad \forall\, \varphi \in L^\infty.$$

By (4) and (6):

$$(7) \qquad \int |f(x)\,\varphi(x)|\,dx \leq \|F\|_{B'} \, \|\varphi\|_B \quad , \forall\, \varphi \in L^\infty.$$

By (4) and by the definition of B,

$$(8) \qquad \lim_{n=\infty} \|T_n \varphi\| = \|\varphi\| \qquad \forall\, \varphi \in B$$

By (4), (7) and (8),

$$\int |f(x)\,\varphi(x)|\,dx \leq \|F\|_{B'} \, \|\varphi\|_B \qquad \forall\, \varphi \in B$$

By assumption (5)

$$(9) \qquad \lim_{n=\infty} F(T_n \varphi) = F(\varphi) \qquad \forall\, \varphi \in B$$

and we obtain

$$(10) \qquad F(\varphi) = \int f(x)\,\varphi(x)\,dx \qquad \forall\, \varphi \in B.$$

In order to prove the statement about multipliers ρ, define