Jan. 8, 1963.
Dear Jacques:
Just some notes!
A. B.

Notes on Harmonic Analysis

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Def. The lower uniform density (l.u.d.) of a $\Lambda \subset R$ is

$$(1) \quad \text{l.u.d}\,\Lambda = \lim_{r\to\infty} \frac{1}{r} \left\{ \inf_x \sum_{\substack{x \le \lambda < x+r \\ \lambda \in \Lambda}} 1 \right\}$$

(the limit does always exists ~~because~~ because $\{\ \}$ is a superaddition function of r)

Theorem I. Let E_a denote the set of entire functions $f(z)$ such that

$$(2) \quad \limsup_{|z|\to\infty} \frac{\log|f(z)|}{|z|} \le a\pi \quad (a>0 \text{ given})$$

and let Λ be a uniformly discrete set on R. The necessary and sufficient condition that there exists a finite constant $K = K(\Lambda, a)$ such that ~~the~~ $f \in E_a$, $|f(x)| \le 1$ on Λ, imply $|f(x)| \le K$ for all real x, is that

$$(3) \quad \text{l.u.d.}\,\Lambda > a$$

We shall only need the sufficiency of (3). The proof requires these definitions. Set $[x, \Lambda] = \text{dist.}(x, \Lambda)$ and observe that $[x, \Lambda]$ is unif. Lip. 1.

Def. A sequence of closed sets $\{\Lambda_n\}_1^\infty$ converges weakly to Λ if $[x, \Lambda_n] \to [x, \Lambda]$ pointwise (and thus uniformly on compact sets)

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Def. A sequence of closed sets $\{\Lambda_n\}_1^\infty$ converges weakly to Λ if $[x, \Lambda_n] \to [x, \Lambda]$ pointwise (and thus uniformly on compact sets)

$\{\Lambda_n\}$ converges to the empty set if $[x, \Lambda_n] \to \infty$.

Obvious properties: Each sequence $\{\Lambda_n\}_1^\infty$ contains a subsequence converging weakly to a set Λ (empty or not). Let Λ^τ denote the τ-translate of Λ and let $W(\Lambda)$ be the collection of all weak limits of translates of Λ. For each $\Lambda_0 \in W(\Lambda)$

(4) $$\text{l.u.d. } \Lambda_0 \geq \text{l.u.d. } \Lambda$$

Proof of the sufficiency of (3). Assume that our conditions imply that

(5) $$\|f\| = \sup_{-\infty < x < \infty} |f(x)| < \infty$$

An application of the Phragmén-Lindelöf principle to the half planes $y > 0$ and $y < 0$ yields

$$|f(x+iy)| \leq \|f\| \, e^{a\pi|y|}.$$

If therefore (5) holds for $f \in E_a$, ~~with~~ f bounded on Λ, then these functions form a Banach space with norm $\|f\|$. But

$$\|f\|_\Lambda \equiv \sup_{x \in \Lambda} |f(x)| \leq \|f\|.$$

By the closed graph theorem there exist a constant K such that

$$\|f\|_\Lambda \leq \|f\| \leq K \|f\|_\Lambda$$

It is therefore sufficient to prove (5).

Assume $\limsup_{x=\infty} |f(x)| = \infty$. We have

$$\mathrm{Re}\{z \log z\} = \begin{cases} x \log x & \text{on } R^+ \\ -\frac{\pi}{2}|y| & \text{on imag. axis.} \end{cases}$$

Define for $0 < \varepsilon$

(7) $$\sup_{x>0}\{\log|f(x)| - \varepsilon x \log x\} = c_\varepsilon$$

Then $\liminf_{\varepsilon \downarrow 0} c_\varepsilon = \infty$. Let x_ε denote the largest number x such that

(8) $$\log|f(x)| - \varepsilon x \log x = c_\varepsilon$$

Clearly: $\lim_{\varepsilon \downarrow 0} x_\varepsilon = \infty$. Set

(9) $$g_\varepsilon(z) = f(z) e^{-\varepsilon z \log z - c_\varepsilon}, \quad x > 0$$

Then

(10) $$|g_\varepsilon(x)| \le 1 \qquad x \in R^+$$

(11) $$|g_\varepsilon(x)| \le e^{-c_\varepsilon}, \quad x \in \Lambda,\ x \ge 1.$$

For each $b > a$,

$$|f(iy)| \le M(b) e^{b\pi|y|}. \qquad (M(b) = M < \infty)$$

Hence

(12) $$|g_\varepsilon(iy)| \le M e^{(b+\frac{\varepsilon}{2})\pi|y|}$$

By (10) and (12) it follows that: (Phragmén-Lindelöf in two quadrants)

(13) $$|g_\varepsilon(x+iy)| \le M e^{(b+\frac{\varepsilon}{2})\pi|y|} \qquad x \ge 0$$

and this implies

(14) $$|g_\varepsilon'(x)| \leq \text{const.} \quad (x \geq 1, \quad \varepsilon > 0).$$

There exists now a sequence $\{\varepsilon_\nu\}$, $\varepsilon_\nu \to 0$, such that the functions $g_{\varepsilon_\nu}(z - x_{\varepsilon_\nu})$ converges to an entire function $h(z)$, which by virtue of (13) and the fact that $x_{\varepsilon_\nu} \to \infty$, must belong to E_b. We may also assume that the translates $\Lambda^{x_{\varepsilon_\nu}}$ converges weakly to some set $\Lambda_0 \in W(\Lambda)$. By the definition of x_ε it follows that $|h(0)| = 1$. By the equicontinuity (14) it follows that $h = 0$ on Λ_0. Summing up: $h(z)$ is an entire function of exponential type $\leq b\pi$ vanishing on a uniformly discrete set Λ_0 with l.u.d. $\Lambda_0 > a$. It is well known that this implies $h \equiv 0$ if (according to (4)) $b <$ l.u.d. Λ_0.
But we can chose b so that $a < b <$ l.u.d. Λ_0. This contradiction to $|h(0)| = 1$ proves that f is bounded on R. The proof of the sufficiency of (3) is complete.

Application: Let λ' and λ'' be neg. def. functions on J such that $\lambda'(n), \lambda''(n) \neq 0$ for $n \neq 0$. Let F be a closed set $\subset [-a\pi, a\pi]$, $a < 1$ and let T be a distribution on F with

$$\|T\|^2 = \sum_{n \neq 0} |\hat{T}(n)|^2 \frac{\lambda'(n)}{\lambda''(n)} < \infty$$

$\hat{T}(n)$ is the restriction to J of an entire function $f(z) \in E_a$. We also know that

(16) $$\frac{\lambda'(n)}{\lambda''(n)} \geq \frac{\delta}{n^2}$$

on a set of integers Λ such that l.u.d. $\Lambda > a$ if δ is sufficiently small. By (15)

(17) $$|f(x)| \leq \|T\| \cdot \sqrt{\tfrac{1}{\delta}}\ |x| \quad \text{on } \Lambda$$

By a modification of the proof of the previous theorem it follows that K exists such that (17) implies

(18) $$|f(n)| \leq \frac{\|T\|\, K}{\sqrt{\delta}}\ |n|, \quad n \neq 0$$

~~Entire function $f \in E_a$ satisfying (18) forms a compact set (if $\|T\|$ bounded)~~, or equivalently

(19) $$|f(n)| \leq \|T\|\, K'\,(1+|n|) \quad \text{all } n \in J.$$

By this we conclude that if $\{T_n\}$ is a Cauchy sequence $\subset \mathcal{H}_F$, then the entire functions $\hat{T}_n(z)$ converges to an entire function which is the transform of a (15) distribution with support in F. $\mathcal{H}_F$ is therefore complete.

Good Night!