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Def. ... $\int_a^b \frac{u(y)\,dy}{|x-y|^\alpha}$ $\quad 0<\alpha<1$

... $\frac{d}{dx}$ ...

For functions u vanishing off (a,b)

Δ is the Laplace operator in a Dir. space $\mathcal{D}$

with square norm

$$\|u\|^2 = \int_a^b\int_a^b \frac{u'(x)\,u'(y)}{|x-y|^\alpha}\,dx\,dy$$

(If u is defined as $=0$ off (a,b) then

$$\|u\|^2 = c_\alpha \int_{-\infty}^{\infty} |\hat{u}(t)|^2 |t|^{1+\alpha}, \quad (\lambda(t) = |t|^{1+\alpha} \text{ is}$$

neg. def.) . Let

$$K(\xi_1,\xi_2) = c_{1,1}\,\xi_1^2 + 2c_{1,2}\,\xi_1\xi_2 + c_{2,2}\,\xi_2^2$$

be a pos. def. quadratic form . Then

$\|u\|_{H_0}^2 = \|u\|^2 + K(u(a),u(b))$ is the square norm

in a certain Hilbert space $\mathcal{D}_0$.

Define $Au = A(u, x) = \int_a^b \frac{u(y)\,dy}{|x-y|^\alpha}$ $\quad 0 < \alpha < 1$

$$\Delta u = -\frac{d}{dx} A(u, x)$$

For functions u vanishing off (a, b)

Δ is the Laplace operator in a Dir. space $\mathcal{D}$

with square norm

$$\|u\|^2 = \int_a^b \int_a^b \frac{u'(x)\,u'(y)}{|x-y|^\alpha}\,dx\,dy$$

(If u is defined as $= 0$ off (a, b) then

$\|u\|^2 = c_\alpha \int_{-\infty}^{\infty} |\hat{u}(t)|^2 |t|^{1+\alpha}$, $\left(\lambda(t) = |t|^{1+\alpha}\right.$ is

neg. def.) . Let

$$K(\xi_1, \xi_2) = c_{1,1}\,\xi_1^2 + 2c_{1,2}\,\xi_1\,\xi_2 + c_{2,2}\,\xi_2^2$$

be ~~of~~ a pos. def. quadratic form. Then

$\|u\|_0^2 = \|u\|^2 + K(u(a), u(b))$ is the square norm

in a certain Hilbert space $\mathcal{D}_0$.

$$(u,v)_0 = \int_a^b\int_a^b \frac{u'(x)v'(y)\,dx\,dy}{|x-y|^\alpha} + c_{1,1}\, u(a)v(a) +$$

$$+ c_{1,2}\, u(a)v(b) + c_{1,2}\, u(b)v(a) + c_{2,2}\, u(b)v(b)$$

$$\int_a^b\int_a^b = \int_a^b v'(y)\, A(u,y)\, dy = v(b)A(u,b) - v(a)A(u,a)$$
$$+ \int_a^b v(y)\, A(u,y)\, dy$$

$\Rightarrow$

$$(u,v)_0 = \int_a^b v(x)\, A(u,x)\, dx + v(b)\left\{A(u,b) + c_{1,2}\, u(a) + c_{2,2}\, u(b)\right\} +$$
$$+ v(a)\left\{-A(u,a) + c_{1,1}\, u(a) + c_{1,2}\, u(b)\right\}$$

Consider the problem

$$\cancel{\inf_{u\in\mathcal{D}_0}\left\{\lambda\|u\|_0^2 + \int_a^b (u-f)^2\,dx\right\}. \quad \lambda>0}$$

$$\cancel{\lambda(u,v)_0 + \int_a^b (u-f)\,v\,dx = 0 \quad \text{for } v\in\mathcal{D}_0 \Rightarrow}$$

$$\inf_{u\in\mathcal{D}_0}\left\{\lambda\|u\|^2 + K(u(a),u(b))\right.$$

(meas.)

$$\inf_{u\in\mathcal{D}_0}\left\{\lambda\|u\|_0^2 + \int_a^b (u-f)^2\,dx\right\}, \quad \lambda>0, \quad f\in L^\infty(a,b)$$

Minimizing u exists and

$$\lambda(u,v)_0 + \int_a^b (u-f)\,v\,dx = 0, \quad v\in\mathcal{D}_0 \Rightarrow$$

$$\Rightarrow \int_a^b v(x)\{\lambda\Delta u + u - f\}\,dx + \lambda v(b)\{\quad\} + \lambda v(a)\{\quad\} = 0$$

$$\Rightarrow \begin{cases} \lambda\Delta u + u - f = 0 & \text{on } (a,b) \\ A(u,b) + c_{12}\,u(a) + c_{22}\,u(b) = 0 & \\ -A(u,a) + c_{11}\,u(a) + c_{12}\,u(b) = 0 & \end{cases}$$

For which quadr. forms. Is $\mathcal{D}_0$ a Dir. space?